The resulting wave equation for $\bar{E}$ then becomes

$$\nabla^2 \bar{E} + \omega^2 \mu\epsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \bar{E} = 0, \tag{1.51}$$

where we see a similarity with (1.42), the wave equation for $\bar{E}$ in the lossless case. The difference is that the quantity $k^2 = \omega^2\mu\epsilon$ of (1.42) is replaced by $\omega^2\mu\epsilon[1 - j(\sigma/\omega\epsilon)]$ in (1.51). We then define a *complex propagation constant* for the medium as

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \tag{1.52}$$

where α is the *attenuation constant* and β is the *phase constant.* If we again assume an electric field with only an $\hat{x}$ component and uniform in x and y, the wave equation of (1.51) reduces to

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0, \tag{1.53}$$

which has solutions

$$E_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}. \tag{1.54}$$

The positive traveling wave then has a propagation factor of the form

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z},$$

which in the time domain is of the form

$$e^{-\alpha z} \cos(\omega t - \beta z).$$

We see that this represents a wave traveling in the $+z$ direction with a phase velocity $v_p = \omega/\beta$, a wavelength $\lambda = 2\pi/\beta$, and an exponential damping factor. The rate of decay with distance is given by the attenuation constant, α. The negative traveling wave term of (1.54) is similarly damped along the $-z$ axis. If the loss is removed, $\sigma = 0$, and we have $\gamma = jk$ and $\alpha = 0$, $\beta = k$.

As discussed in Section 1.3, loss can also be treated through the use of a complex permittivity. From (1.52) and (1.20) with $\sigma = 0$ but $\epsilon = \epsilon' - j\epsilon''$ complex, we have that

$$\gamma = j\omega\sqrt{\mu\epsilon} = jk = j\omega\sqrt{\mu\epsilon'(1 - j\tan\delta)}, \tag{1.55}$$

where $\tan\delta = \epsilon''/\epsilon'$ is the loss tangent of the material.

The associated magnetic field can be calculated as

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{-j\gamma}{\omega\mu}(E^+ e^{-\gamma z} - E^- e^{\gamma z}). \tag{1.56}$$

The intrinsic impedance of the conducting medium is now complex,

$$\eta = \frac{j\omega\mu}{\gamma}, \tag{1.57}$$

but is still identified as the wave impedance, which expresses the ratio of electric to magnetic field components. This allows (1.56) to be rewritten as

$$H_y = \frac{1}{\eta}(E^+ e^{-\gamma z} - E^- e^{\gamma z}). \tag{1.58}$$

Note that although η of (1.57) is, in general, complex, it reduces to the lossless case of $\eta = \sqrt{\mu/\epsilon}$ when $\gamma = jk = j\omega\sqrt{\mu\epsilon}$.