14. Prove, by induction or otherwise, that

$$\sum_{k=1}^{n} 3^k(2k+5) = 3^{n+1}(n+2) - 6. \quad \square$$

Induction solution

Let $S_n = \sum_{k=1}^{n} 3^k(2k+5)$

and $f(n) = 3^{n+1}(n+2) - 6$.

Then $S_1 = 3(2+5) = 21$

and $f(1) = 3^2(3) - 6 = 21$.

Hence result is true when $n = 1$.

Assume $S_m = f(m)$ for some integer m.

Then
$$\begin{aligned} S_{m+1} &= S_m + 3^{m+1}(2m+7) \\ &= 3^{m+1}(m+2) - 6 + 3^{m+1}(2m+7) \\ &= 3^{m+1}(3m+9) - 6 \\ &= 3^{m+2}(m+3) - 6 = f(m+1). \end{aligned}$$

So if $S_m = f(m)$ we may deduce that $S_{m+1} = f(m+1)$.

But $S_1 = f(1)$ and so $S_2 = f(2)$.

This in turn implies that $S_3 = f(3)$ and so on for any integer n. Hence the result is true for all integer n. ■

Alternative solution

We use the method of differences F2.

Let $f(k) = 3^{k+1}(k+2) - 6$. Then

$$\begin{aligned} f(k) - f(k-1) &= 3^{k+1}(k+2) - 6 - 3^k(k+1) + 6 \\ &= 3^k(3k+6-k-1) \\ &= 3^k(2k+5). \end{aligned}$$

Hence $\sum_{k=1}^{n} 3^k(2k+5) = S_n$, where

$$\begin{aligned} S_n &= [f(1) - f(0)] + [f(2) - f(1)] + \cdots \\ &\quad + [f(n) - f(n-1)] \\ &= f(n) - f(0) \\ &= 3^{n+1}(n+2) - 6 - 3(2) + 6 \\ &= 3^n(n+2) - 6. \end{aligned}$$