48. Find $\mathcal{L}\{\sin\sqrt{t}\}$.

Method 1, using series.

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

Then the Laplace transform is

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - \frac{\Gamma(9/2)}{7!\,s^{9/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{(1/2^2 s)^2}{2!} - \frac{(1/2^2 s)^3}{3!} + \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/2^2 s} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}$$

Method 2, using differential equations.

Let $Y(t) = \sin\sqrt{t}$. Then by differentiating twice we find

$$4tY'' + 2Y' + Y = 0$$

Taking the Laplace transform, we have if $y = \mathcal{L}\{Y(t)\}$

$$-4\frac{d}{ds}\{s^2 y - s\,Y(0) - Y'(0)\} + 2\{s\,y - Y(0)\} + y = 0$$

or

$$4\,s^2 y' + (6\,s - 1)y = 0$$

Solving,

$$y = \frac{c}{s^{3/2}}\,e^{-1/4s}$$

For small values of t, we have $\sin\sqrt{t} \sim \sqrt{t}$ and $\mathcal{L}\{\sqrt{t}\} = \sqrt{\pi}/2s^{3/2}$. For large s, $y \sim c/s^{3/2}$. It follows by comparison that $c = \sqrt{\pi}/2$. Thus

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}$$

49. Find $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Let $F(t) = \sin\sqrt{t}$. Then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$, $F(0) = 0$. Hence by Problem 48,

$$\mathcal{L}\{F'(t)\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\,f(s) - F(0) = \frac{\sqrt{\pi}}{2\,s^{1/2}}\,e^{-1/4s}$$

from which

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}\,e^{-1/4s}$$

The method of series can also be used [see Problem 175(*b*)].

50. Show that

$$\mathcal{L}\{\ln t\} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where $\gamma = .5772156\ldots$ is *Euler's constant.*

We have

$$\Gamma(r) = \int_0^\infty u^{r-1}\,e^{-u}\,du$$