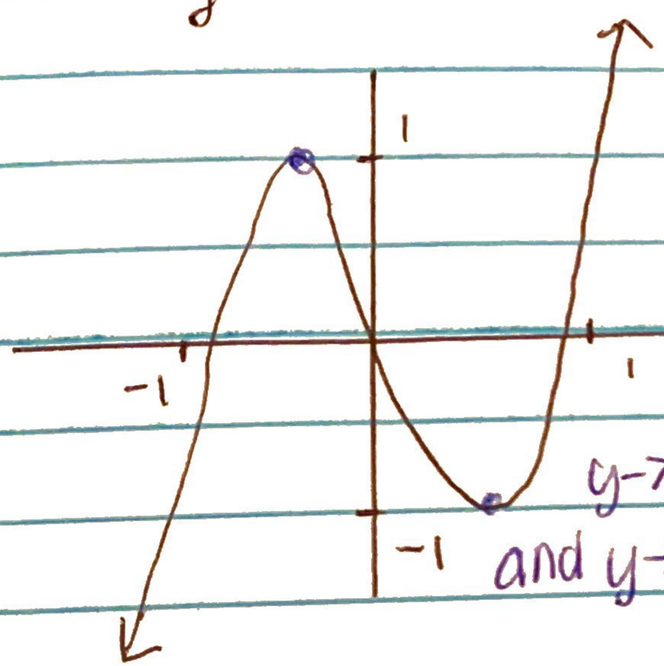
8a $y = 4x^3 - 3x$



onto but not one-to-one since $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$

~~The image of $F^{-1}(y)$ does not contain the portion enclosed in $-\frac{1}{2} < x < \frac{1}{2}$~~

~~since $\frac{dy}{dx} = 12x^2 - 3 > 0$ when $x^2 \leq \frac{1}{4}$.~~

~~$\{x^2 \leq \frac{1}{4}\}$ is the singular set, which is not in the image of F^{-1}. The inverse is not defined for $\{-1 < y < 1\}$. There are two inverses, for $x > \frac{1}{2}$ and $x < \frac{1}{2}$~~

$Dy = 12x^2 - 3$

$12x^2 = 3$

$x^2 = \frac{1}{4}$

$\Sigma = \{x = \pm\frac{1}{2}\}$

$F(\Sigma) = \{y = 1, -1\}$

There are 3 inverses, defined everywhere except $y = \pm 1$.

$\Sigma = \{x = \pm\frac{1}{2}\}$ is not in the image of $F(y)$.

8a F^{-1}:

Trigonometric solution of the cubic trick.

$$y = 4x^3 - 3x$$

$$\Rightarrow -y + y = 4x^3 + 0x^2 - 3x - y$$

$$\Rightarrow 0 = 4x^3 + 0x^2 - 3x - y$$

$$\Rightarrow 0 = x^3 + \underset{A}{0}x^2 \underset{B}{-\tfrac{3}{4}}x \underset{C}{-\tfrac{1}{4}y}$$

$$Q = \frac{3B - A^3}{9} \Rightarrow \frac{-\frac{9}{4}}{9} \Rightarrow \frac{-1}{4}$$

$$R = \frac{9AB - 27C - 2A^3}{54} \Rightarrow \frac{27\left(\frac{y}{4}\right)}{54} \Rightarrow \frac{1}{8}y$$

$$D = Q^3 + R^2 \Rightarrow \frac{-1}{64} + \frac{y^2}{64}$$ Assume $D<0$ and 3 real roots

$$\cos(\theta) = \frac{R}{\sqrt{-Q^3}} \Rightarrow \frac{1}{8}y \, \frac{1}{(1/64)^{1/2}} \Rightarrow \frac{8}{8}y = y \Rightarrow \theta = \arccos y$$

$$x_1 = 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) - \frac{A}{3} \Rightarrow x = \cos\left(\frac{\arccos y}{3}\right) \quad \tfrac{1}{2} \leq x \leq 1$$

$$x_2 = 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \Rightarrow x = \cos\left(\frac{\arccos y}{3} + \frac{2\pi}{3}\right) \quad -1 \leq x \leq -\tfrac{1}{2}$$

$$x_3 = 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) \Rightarrow x = \cos\left(\frac{\arccos y}{3} + \frac{4\pi}{3}\right) \quad -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}$$

These inverses are only defined for values of y between 1 and -1 due to the arccos term being undefined for $|y| > 1$. However there are more ways to derive F^{-1} that are defined for the complement of the singular set.