

ORBITAL MECHANICS NOTES ME230

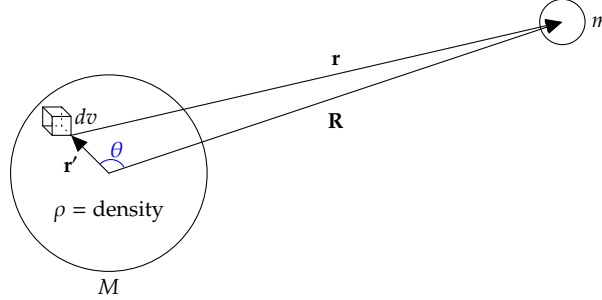
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Gravity

Newton's Law of Gravitation is $F = -\frac{Gm_1m_2}{r^2}$ and gravitational potential is $V = -\frac{Gm_1m_2}{r}$.



From the figure above, we can write the differential potential energy due to a differential volume, dV . Let $dV = -\frac{GMm}{r}$ and $dM = \rho dV$ where ρ is constant. Then

$$\begin{aligned} V &= - \int_V \frac{GdMm}{r} &= -Gm \int_V \frac{\rho dV}{r} \\ &= -Gm \iiint \frac{\rho}{r} (r')^2 \sin \theta d\theta d\varphi dr' &= -\rho Gm \iiint \frac{(r')^2 \sin \theta}{r} d\theta d\varphi dr' \end{aligned}$$

The Law of Cosine for figure can be written as $r^2 = R^2 + r'^2 - 2Rr' \cos \theta$ so $r = \sqrt{R^2 + r'^2 - 2Rr' \cos \theta}$. Therefore, we can substitute r into our integral equation.

$$V = -\rho Gm \iiint \frac{(r')^2 \sin \theta}{\sqrt{R^2 + r'^2 - 2Rr' \cos \theta}} d\theta d\varphi dr'$$

where the region of integration is $0 < r' < a$, $0 < \theta < \pi$, and $0 < \varphi < 2\pi$.

$$\begin{aligned} V &= -\rho Gm \iiint \frac{(r')^2 \sin \theta}{R \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}} d\theta d\varphi dr' \\ &= -2\pi\rho GM \iint \frac{(r')^2 \sin \theta}{R \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}} d\theta dr' \end{aligned} \tag{1}$$

From (1), let $u = \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}$ so $du = \frac{r' \sin \theta}{R \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}} d\theta$.

$$d\theta = \frac{R \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}}{r' \sin \theta}$$

Making the substitution with u and du , we now have

$$\begin{aligned} V &= -2\pi\rho GM \iint r' du dr' \\ &= -2\pi\rho GM \int r' \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta} \Big|_0^\pi dr' \\ &= -2\pi\rho GM \int r' \left[\sqrt{1 + \left(\frac{r'}{R}\right)^2 + 2\frac{r'}{R}} - \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}} \right] dr' \end{aligned}$$

Then $V = -\frac{GmM}{R}$.

Now let's take the general case when $\mathbf{R} = \mathbf{x}$. That is, let's look at the gravitational potential for an arbitrary spheroid. Then

$$V(\mathbf{x}) = \frac{GMm}{x} \left[1 + \sum_{n=1}^{\infty} J_n \left(\frac{a_0}{x} \right)^n P_n(\cos \theta) \right]$$

where a_0 is the mean radius of the body M , θ is the angular location of m , J_n is a constant (zonal harmonic), and P_n is the Legendre Polynomials of order " n ". Recall that for the Law of Cosine, we had

$r = \sqrt{R^2 + r'^2 - 2Rr' \cos \theta} = R \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}$. Then we have the generating function:

$$\frac{1}{\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R} \cos \theta}} = \sum_{n=0}^{\infty} J_n \left(\frac{a_0}{x} \right)^n P_n(\cos \theta).$$

