Math 2300. The neighborhood definition of limit
Either the input or the output (or both) can be infinite, in a limit expression, and these have various definitions which may seem confusingly different. However it is possible to look at all these definitions as essentially the same by introducing the concept of "neighborhoods". The idea of a limit is that no matter how close you want the output f(x) to get to the limit value L, if the input x is close enough to a, but not equal to it, then f(x) will be as close to L as you asked. The precise way to say you can get f(x) as close as you want to L, is to say you can get f(x) in every "neighborhood" of L (no matter how small).

This concept is used in everyday English by people who do not think they understand math. A few years ago my friend had a truck for sale, and when someone asked him the price he said "in the neighborhood of 850 to 900 dollars". Then he turned to me and muttered "that's not a very big neighborhood is it?" meaning he had not left the buyer much room to bargain. He was giving the buyer a 50 dollar neighborhood and implied he would have to get in that range with his offer in order to get the truck. I told him we described things exactly the same way in calculus but he looked skeptical.

Now we want to show how to use neighborhoods to define the symbol

$$\lim_{x\to a} f(x) = L$$

in the same way, whether a is a finite real number, or $+\infty$ or $-\infty$, and whether L is a finite real number, or $+\infty$ or $-\infty$.

Definition: $\lim_{x\to a} f(x) = L$, if and only if: for every neighborhood V of L, there is some neighborhood U of a, such that if x is in the domain of f, and in U, but $x \neq a$, then f(x) is in V.

Let's call a, for the moment, the "domain point" and L the "limit value". Then the definition says: for every neighborhood V of the limit value, there is a neighborhood U of the domain point, such that whenever x is in U (and in the domain of f) but $x \neq a$, then f(x) is in V.

The order in which the neighborhoods occur in the definition, limit neighborhood first, domain neighborhood second, is the important thing to keep in mind, not whether the neighborhood is described by a ∂ or an M, or an e or an N. This definition works even if a or L or both represent $+\infty$ or $-\infty$, as long as you know what a neighborhood of ∞ or $-\infty$ is.

Neighborhoods: A neighborhood of a is supposed to be a set U such that any point x near enough to a will be in U. If a is a finite real number this just means that U contains some interval of form $(a-\partial,a+\partial)$. I.e. if U contains this interval, then any point x closer to a than ∂, will be in U.

Notice that an interval like $[a,a+\partial)$ will not do, because no matter how close a point x gets from the left, as long as $x \neq a$ then x is still not in the interval. Of course if the domain of f is a closed interval like [a,b] then the intersection of the neighborhood $(a-\partial,a+\partial)$ of a with the domain will look like $[a,a+\partial)$. In this case this causes no problem. I.e. then any point x of the domain of f which is close enough to a will lie in $[a,a+\partial)$, so relative to the domain of f, the set $[a,a+\partial)$ does function like a neighborhood of a. In general a neighborhood of a point a, relative to a given set S containing a, is the intersection of S with a neighborhood of a.

To define a neighborhood U of $+\infty$, we want any point close enough to ∞ to be in U, i.e. any point large enough should be in U. Thus a neighborhood U of $+\infty$ is any set U containing an interval of form $(M,+\infty)$ where M is a real number.

Now let's translate the neighborhood definition of limit into the usual one for $a = L = +\infty$:
Definition: $\lim_{x\to+\infty} f(x) = +\infty$, if and only if: for every neighborhood V of $+\infty$, there is some neighborhood U of $+\infty$, such that whenever x is in the domain of f and in U, then f(x) is in V.

[Note we do not need to say $x \neq +\infty$, since x cannot equal $+\infty$ and also be in the domain of f.]

Now every neighborhood of $+\infty$ is an interval of form $(M,+\infty)$ so the definition comes out:
Definition: $\lim_{x\to+\infty} f(x) = +\infty$, if and only if: for every real number N, there is some real number M, such that whenever x is in the domain of f and $x > M$, then $f(x) > N$.

Notice here N defines the neighborhood of the limit point $+\infty$ and M defines the neighborhood of the domain point $+\infty$. I keep them straight in my mind by thinking of x as somehow coming before f(x), so when I choose the domain neighborhood as earlier in the alphabet it comes out in alphabetical order at the end of the definition: "if $x > M$, then $f(x) > N$".

Now let's translate the neighborhood definition into the usual one when a is finite and $L = -\infty$.
Definition: $\lim_{x\to a} f(x) = -\infty$, if and only if for every neighborhood V of $-\infty$, there is some neighborhood U of a, such that if x is in the domain of f, and in U, and $x \neq a$, then f(x) is in V.

A neighborhood of $-\infty$ has form $(-\infty,N)$, and a basic neighborhood of a has form $(a-\partial,a+\partial)$ so the definition comes out:
Definition: $\lim_{x\to a} f(x) = -\infty$, if and only if: for every real number N, there is some $\partial > 0$, such that whenever x is in the domain of f and $0 < |x-a| < \partial$, then $f(x) < N$.

Exercise:
Try writing out $\lim_{x\to-\infty} f(x) = -\infty$, and $\lim_{x\to+\infty} f(x) = 0$, using N's and ∂'s.

Taking the "union" or "piecewise join" of two functions.
Notice that the algebraic operations on functions apply to functions that have the same domain, whereas the composition gof applies to functions such that the domain of g contains the <u>values</u> of f. When we invert a function, the new function has domain equal to the set of values of the old function.

There is another way of combining functions, by taking functions whose domains are adjacent intervals and pasting them together, enlarging the domain by using one function on one part of the domain and the other on the other part.

For example, the absolute value function equals x on the domain $x \geq 0$ and equals -x on the domain $x < 0$. A function like the absolute value function which is defined piecewise by patching together linear functions is called "piecewise linear". The "greatest integer" , or "rounding down" function **[x] =** greatest integer not greater than x, is defined piecewise on each half open interval $[n,n+1)$ by the constant function n. A function like this which is defined piecewise by patching together constant functions is called "piecewise constant". If f is defined on $[a,b]$ and g is defined on $(b,c]$ then we get a new function h defined on $[a,c]$, by patching f and g together piecewise, setting $h(x) = f(x)$ if $a \leq x \leq b$, and $h(x) = g(x)$ if $b < x \leq c$.

If f and g were both continuous, then h is continuous everywhere except possibly at b. Then h is continuous at b if and only if the right hand limit of h at b equals the left hand limit of f at b. If g was continuous on the closed interval $[b,c]$ then h is continuous at b if and only if $f(b) = g(b)$.

Similarly if f and g were both differentiable, [which means f was not only differentiable on (a,b) but also differentiable from the left at b and from the right at a, and g was not only differentiable on (b,c) but also differentiable from the right at b and from the left at c], then h is differentiable at least on $[a,b)$ and $(b,c]$. Then h is also differentiable at b if and only if $f(b) = g(b)$, and the left hand derivative of f at b equals the right hand derivative of g at b, i.e. if and only if $\lim_{x\to b^-} \{f(x)-f(a)\}/(x-a) = \lim_{x\to b^+} \{g(x)-g(a)\}/(x-a)$.

If this happens then the derivative of h at b equals both of these one sided derivatives. A nice example is when $f(x) = -x^2/2$ on $(-\infty,0]$ and $g(x) = x^2/2$ on $[0,+\infty)$. Then f and g patch together to give a function h which is differentiable on all of $(-\infty,+\infty)$ and has derivative $h'(x) = |x|$, the absolute value function. This shows that just because the absolute value function does not <u>have</u> a derivative at 0, that does not mean it cannot <u>be</u> a derivative at 0.