

Differential Calculus Topics

Limits,
Continuous Functions,
Derivatives,
Local and global extrema,
1st derivatives and increasing or decreasing functions,
2nd derivatives and concavity,
MMV, MVT, IVT,
Applied Max Min Problems,
Implicit derivatives and Related Rates,
Asymptotes and Graphing,
Antiderivatives and applications.

LIMITS:

Very roughly speaking: If L is some finite number, then to say $\lim_{x \rightarrow a} f(x) = L$, means that if you want $f(x)$ very close to L , all you have to do is make x close enough to a .

$\lim_{x \rightarrow a^+} f(x) = L$, means if you want $f(x)$ very close to L , all you have to do is make x close enough to a , and $x > a$.

$\lim_{x \rightarrow a^-} f(x) = L$, means that if you want $f(x)$ very close to L , all you have to do is make x very close to a , and $x < a$.

$\lim_{x \rightarrow a} f(x) = +\infty$, means that if you want $f(x)$ very large and positive, all you have to do is make x very close to a .

$\lim_{x \rightarrow a} f(x) = -\infty$, means that if you want $f(x)$ very large and negative, all you have to do is make x is very close to a .

$\lim_{x \rightarrow +\infty} f(x) = L$, means that if you want $f(x)$ very close to L , all you have to do is make x is very large and positive.

$\lim_{x \rightarrow -\infty} f(x) = L$, means that if you want $f(x)$ very close to L , all you have to do is make x is very large and negative.

Get the pattern?

A fundamental principle: if two functions have the same values for all x near a , then they have the same limit “at a ”, i.e. as $x \rightarrow a$, (even if they do not have the same value at a).

For example, the two functions x^2/x , and x have the same values except at 0 , hence have the same

limit at 0. Since obviously x gets near zero as x gets near zero, and since $x^2/x = x$ for all $x \neq 0$, we see that x^2/x also must get near zero as x gets near zero. I.e. the fact that $\lim_{x \rightarrow 0} x = 0$, implies that also $\lim_{x \rightarrow 0} x^2/x = 0$.

Some useful Limits:

$\lim_{x \rightarrow +\infty} 1/x = 0^+$; (this means $1/x$ approaches 0 through positive values.)

$\lim_{x \rightarrow -\infty} 1/x = 0^-$;

$\lim_{x \rightarrow 0^+} 1/x = +\infty$;

$\lim_{x \rightarrow 0^-} 1/x = -\infty$;

$\lim_{x \rightarrow a^+} 1/(x-a) = +\infty$;

$\lim_{x \rightarrow a^-} 1/(x-a) = -\infty$;

$\lim_{x \rightarrow a^+} 1/(x-a)^2 = +\infty$;

$\lim_{x \rightarrow a^-} 1/(x-a)^2 = +\infty$.

Continuous Functions: A function f is continuous at c iff: c is in the domain of f , and $\lim_{x \rightarrow c} f(x) = f(c)$. (In particular $\lim_{x \rightarrow c} f(x)$ is a finite number.)

All polynomials, all n th root functions where n is odd, and the functions \sin , \cos and e^x are continuous at all real numbers.

The “positive n th root” functions where n is even, are continuous on their domain, the non negative reals.

The function $\ln(x)$ is continuous on its domain, the positive real numbers.

Sums, differences, products, and compositions of continuous functions are continuous wherever defined.

Quotients of continuous functions are also continuous where defined, i.e. where the tops and bottoms are both defined, and the bottoms are not zero.

Easy limits: A very useful fact about continuous functions is that once you know a function is continuous, to find its limit at a point in its domain, you only have to evaluate it there: i.e. its limit and its value are the same.

Eg. The polynomial $4x^3 + 5x - 7$ is continuous everywhere,
so $\lim_{x \rightarrow 2} 4x^3 + 5x - 7 = 4(2)^3 + 5(2) - 7 = 35$.

There are two fundamental theorems about continuous functions: MMV and IVT:

MMV: (Maximum/Minimum Value theorem): If f is continuous on the closed bounded interval $[a,b]$, then f takes on both a global maximum and a global minimum on $[a,b]$. I.e. there is some c and some d in $[a,b]$, not necessarily unique, such that for every other x in $[a,b]$ we have

$$f(c) \leq f(x) \leq f(d).$$

In particular, a function that is continuous on a closed bounded interval cannot become infinitely large there, either positively or negatively, since it never gets smaller than some finite minimum value $f(c)$, nor greater than some finite maximum value $f(d)$.

IVT: (Intermediate value theorem): If f is continuous on an interval that contains two points a and b , and if L is any number such that $f(a) < L < f(b)$, then there is at least one point c with $a < c < b$, and $f(c) = L$.

In particular, if $f(a) < 0$, and $f(b) > 0$, and if f is continuous on the interval $[a,b]$, then at some point c between a and b , $f(c)$ must equal 0. Thus to prove that the equation $f(x) = 0$ has a solution for some x in the interval (A,B) , it suffices to find a and b such that $A \leq a < b \leq B$, and $f(a) < 0$, $f(b) > 0$, and f is continuous on $[a,b]$.

For example to show that $e^x + x^5 = 0$ has solution for x in $(-1,0)$, note that for $x=0$, we have $e^0 + 0^5 = 1 > 0$, and for $x = -1$, we have $e^{-1} + (-1)^5 < 0$, [because $e > 2$, so $e^{-1} = 1/e < 1/2$, and $(-1)^5 = -1$, so $e^{-1} + (-1)^5 < 1/2 - 1 = -1/2 < 0$. Moreover $e^x + x^5$ is continuous everywhere, so the IVT applies to prove there is a solution of $e^x + x^5 = 0$ in the interval $(-1,0)$.

The derivative of a function f at a point a in domain f may or may not exist, even if f is continuous at a .

The derivative of f at a exists iff the limit: $\lim_{x \rightarrow a} \{f(x) - f(a)\} / (x-a)$ exists and is finite.

In that case this finite limit is denoted by $f'(a)$.

Geometrically this means that the graph of f has a unique tangent line at the point $(a, f(a))$, which lies directly above the point $(a,0)$ on the x axis. The slope of that tangent line is equal to $f'(a)$, by definition.

Thus limits and derivatives give us a precise way to define tangent lines to very general curves.

(After reading Euclid more carefully, I have learned that actually he characterized tangent lines essentially the same as Newton, as a line that is a limit of secants. I.e. Euclid defined the tangent line to a circle as a line that meets but does not cross it, a definition that works for all convex curves. Then he proves in Prop. III.16 that it is the unique line such that every other line makes an angle that admits an intermediate secant, hence the tangent line is the unique kline that is approximated by secants to any degree of accuracy.)

Since the tangent line to the graph of f at the point $(a, f(a))$, certainly passes through the point $(a, f(a))$, and has slope $f'(a)$, this line has equation $y - f(a) = f'(a)(x - a)$.

Some important derivatives, and general derivative formulas:

$$(f+g)'(a) = f'(a) + g'(a);$$

$(cf)'(a) = cf'(a)$ if c is a constant;
 $c' = 0$ if c is a constant;
 $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;
 $(f/g)'(a) = \{f'(a)g(a) - f(a)g'(a)\} / g^2(a)$.

$(x^n)' = nx^{n-1}$; i.e. if $f(x) = x^n$, then $f'(a) = na^{n-1}$;
 This rule is true not only for n a positive integer but for n any constant:

$(5x^8)' = 40x^7$,
 $(3x^{-1/3})' = -x^{-4/3}$;
 $\sin'(a) = \cos(a)$;
 $\cos'(a) = -\sin(a)$;
 $\tan'(a) = \sec^2(a)$;
 $\sec'(a) = \sec(a)\tan(a)$;
 $\cot'(a) = -\csc^2(a)$;
 $\cot'(a) = -\csc(a)\cot(a)$;
 $(e^x)' = e^x$; i.e. if $f(x) = e^x$ then $f'(a) = e^a$;
 $\ln'(a) = 1/a$, for $a > 0$.

The most important rule for derivatives is the rule for differentiating compositions, the **“chain rule”**: i.e. If $(f \circ g)(x) = f(g(x))$, then $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Eg. $\{\sin(x^4)\}' = (\cos(x^4)) (4x^3)$, and
 $\{\sin^4(x)\}' = (4\sin^3(x))(\cos(x))$.
 $\{e^{\sin(x)}\}' = e^{\sin(x)} \cos(x)$;
 $\{e^{6x}\}' = e^{6x} (6) = 6 e^{6x}$.
 $\{\tan(\tan(x^2))\}' = \{\sec^2(\tan(x^2))\} [\sec^2(x^2)] (2x)$.

A function is called differentiable if it has a derivative. The rules above hold whenever they make sense, i.e. sums, differences, products, and compositions of differentiable functions are differentiable wherever defined. Quotients of differentiable functions are also differentiable where defined, i.e. where the tops and bottoms are both defined and differentiable and the bottoms are not zero.

There is one fundamental theorem relating the property of differentiability with that of continuity: it says that it is "harder" to be differentiable than to be continuous, i.e. **every differentiable function is also continuous, but not conversely**; i.e. there exist continuous functions which are not differentiable. We saw that $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$, and the same holds for $f(x) = x^{1/3}$. Recall that $|x|$ fails to have a derivative at 0, because the expression $|x|/x$ has different one sided limits as x approaches zero, while $x^{1/3}$ fails to have a derivative because the expression $x^{1/3} / x$ has infinite limit as x approaches zero.

Derivatives and local extrema:

Because the derivative of f is the slope of the graph of f , it is not hard to believe that when the derivative of f is positive on the interval (a,b) then f is increasing everywhere on this interval,

although that is a little hard to actually prove. Similarly if the derivative of f is negative on (a,b) then f is decreasing everywhere on (a,b) . It is also true that if the derivative of f is zero everywhere on (a,b) then f is neither increasing nor decreasing anywhere on (a,b) but f is constant on (a,b) . What happens if the derivative of f is only zero at one point of (a,b) ? We really cannot say anything much about f just from that information, but we can say that this is an interesting point and should be looked at more closely.

We can get a true statement by looking at a previous statement backwards: i.e. if knowing that the derivative is always positive or always negative tells us that f is always increasing or always decreasing, then in case f is not always either increasing or decreasing on (a,b) , for example if f has a maximum at a point c in (a,b) then the derivative of f cannot always have the same sign on (a,b) . Moreover it turns out that the derivative of f always has the intermediate value property, so if it does not always have the same sign, then it must be zero somewhere. In fact, if f has a maximum at c in (a,b) , and if f has a derivative at c , then the derivative of f is actually equal to zero at c . [It is important here that c is not an endpoint of the interval; indeed if the maximum of f occurs at an endpoint we cannot expect the derivative necessarily to be zero there.]

Putting these remarks together with the MMV theorem, we get the following **procedure for finding where the maxima and minima of continuous functions occur on closed intervals**:

To find the point of $[a,b]$ where a continuous function f has its absolute (i.e. global) maximum, just examine the values of f at the endpoints a,b and at the “critical points” of f in (a,b) , i.e. those points where f either has derivative equal to zero, or where the derivative fails to exist. Any one of these at which f has the greatest value is a point where the maximum occurs for the whole interval. (There is only one maximum value but it may occur at more than one point.) Similarly for finding the minimum of f over the entire interval, i.e. it occurs at any point among this same collection, the endpoints and the critical points, at which f has the lowest value.

When seeking where f has its global max or min over an open interval we are faced with the additional complication that f may not have one or the other, or may have neither. The following tests are often useful:

The limit test: If the limits of $f(x)$ at both ends of the interval (a,b) are $+\infty$, and if f is continuous on (a,b) then f does have a minimum on (a,b) . This minimum occurs among the critical points of f in (a,b) , i.e. at any one where f has the lowest value. A similar criterion gives a maximum, if the limits are $-\infty$.

The zeroth derivative test: Assume f has only one critical point on (a,b) , say at c . Then f has its global minimum at c if f has a greater value at some point to the left of c and also at some point to the right of c . Similarly, if c is the only critical point of f on (a,b) , f has its global maximum at c if f has a smaller value at some point to the left of c and also at some point to the right of c .

The first derivative test: Assume f has only one critical point on (a,b) , say at c . Then f has its global minimum at c if f has negative derivative at some point to the left of c and positive derivative at some point to the right of c . Similarly, if c is the only critical point of f on (a,b) , f has its global maximum at c if f has positive

derivative at some point to the left of c and negative derivative at some point to the right of c .

The second derivative test: Assume f has only one critical point on (a,b) , say at c . Then f has its global minimum at c if f has positive second derivative at c .

Similarly, if c is the only critical point of f on (a,b) , then f has its global maximum at c if f has negative second derivative at c .

(There are similar tests for local maxima and local minima.)

An inflection point of a graph is a point where the concavity of the graph changes from up to down or from down to up. A graph is concave up on an interval if the graph lies above its tangent lines on that interval, and concave down if the graph lies below the tangent lines.

At an inflection point either the second derivative fails to exist or it is zero. A second order critical point is one where either the second derivative fails to exist or is zero. Thus an inflection point (or flex for short) is always a second order critical point, but not conversely.

(Recall that a point where the graph changes from rising to falling or vice versa is always a critical point, but not conversely.)

One should be able to detect on which intervals a function is rising or falling, and on which intervals a function is concave up or down, by using the first and second derivatives. But in practice one can often save time just by finding the first and second order critical points and plotting them. It is often clear then from their relative positions what the rising and falling behavior of f is, and where the max, min, and flex points are. One usually has also to find the limits of f at the missing endpoints of the domain, including $+\infty$ and $-\infty$.

E.g. if you graph three consecutive critical points $\{a,b,c\}$ and at the middle one the graph is higher than at the other two, then the function is increasing on (a,b) , decreasing on (b,c) , and b is a local maximum. Moreover there is an inflection point somewhere in (a,b) and also somewhere in (b,c) .

I.e. the following facts are very helpful in graphing a differentiable function:

Between two successive critical points the function is always either strictly increasing or strictly decreasing, and there is always at least one flex.

Between two points at which the function has the same height, there is always at least one point where it has either a local max or a local min.

Asymptotes:

The graph of $f(x)$ has a vertical asymptote $x = L$ if $\lim_{x \rightarrow L^+} f(x) = +\infty$, or if $\lim_{x \rightarrow L^-} f(x) = +\infty$, or if $\lim_{x \rightarrow L^+} f(x) = -\infty$, or if $\lim_{x \rightarrow L^-} f(x) = -\infty$.

There is a horizontal asymptote $y = L$ if $\lim_{x \rightarrow +\infty} f(x) = L$, or if $\lim_{x \rightarrow -\infty} f(x) = L$.

The proof of most of the tests for maxs and mins rests on a very important theorem, the MVT.

MVT (Mean Value theorem): If f is continuous on $[a,b]$ and differentiable at least on (a,b) , then there is at least one point c in (a,b) where $f'(c) = \{f(b) - f(a)\} / (b-a)$.

In my opinion **the most important consequence** of this theorem is the following:

Corollary: If f, g are two functions both continuous on $[a,b]$ and both differentiable at least on (a,b) , and if $f'(x) = g'(x)$ for every x in (a,b) , then f and g differ at most by a constant on $[a,b]$; i.e. for some number K , we have $f(x) = g(x) + K$, for every x in $[a,b]$.

The importance of this theorem is that it means if you can figure out the derivative of a formula for something, you are very close to figuring out the formula itself. I.e. you are off at most by a constant.

It turns out for instance that it is easy to find the derivative of a lot of area and volume formulas, and this technique then lets you find the area and volume formulas themselves, by working backwards from their derivatives.

For this reason it becomes useful to practice finding functions whose derivatives are equal to given functions. This process is called antidifferentiation, and the symbol used for antiderivatives is also called an indefinite integral sign, and looks a little like a long S: voila: " \int ".

Thus we have $\int 2x = x^2$, so also $\int x = (1/2)x^2$. and $\int x^2 = (1/3)x^3$. And $\int \cos(x) = \sin(x)$. If you don't believe me, differentiate back and see that you get what you started with. I.e. the derivative of $\sin(x)$ is $\cos(x)$ so one antiderivative of $\cos(x)$ is $\sin(x)$.

Of course another antiderivative of $\cos(x)$ is $\sin(x) + 459$. Try it and see. But by the corollary above we know there are no other possible antiderivatives other than the one we have found except those gotten by adding a constant to ours.

Here are a few more:

$$\int 1/x^2 = -1/x. \quad \int 1/x = \ln(x). \quad \int e^{2x} = (1/2) e^{2x}. \quad \int \tan(x) = \ln(\sec(x)).$$

Next we give some powerful applications of the method of antiderivatives to finding area and volume formulas.

Recall that the secret to finding area under a graph was to show that the derivative of the area function is the height function for the graph. Then the area function is the antiderivative of the height function, and this often lets us guess the area function from the height function. We can find many more formulas, such as volumes, by this method. All we have to do is figure out the derivative of the volume function, and then work backwards.

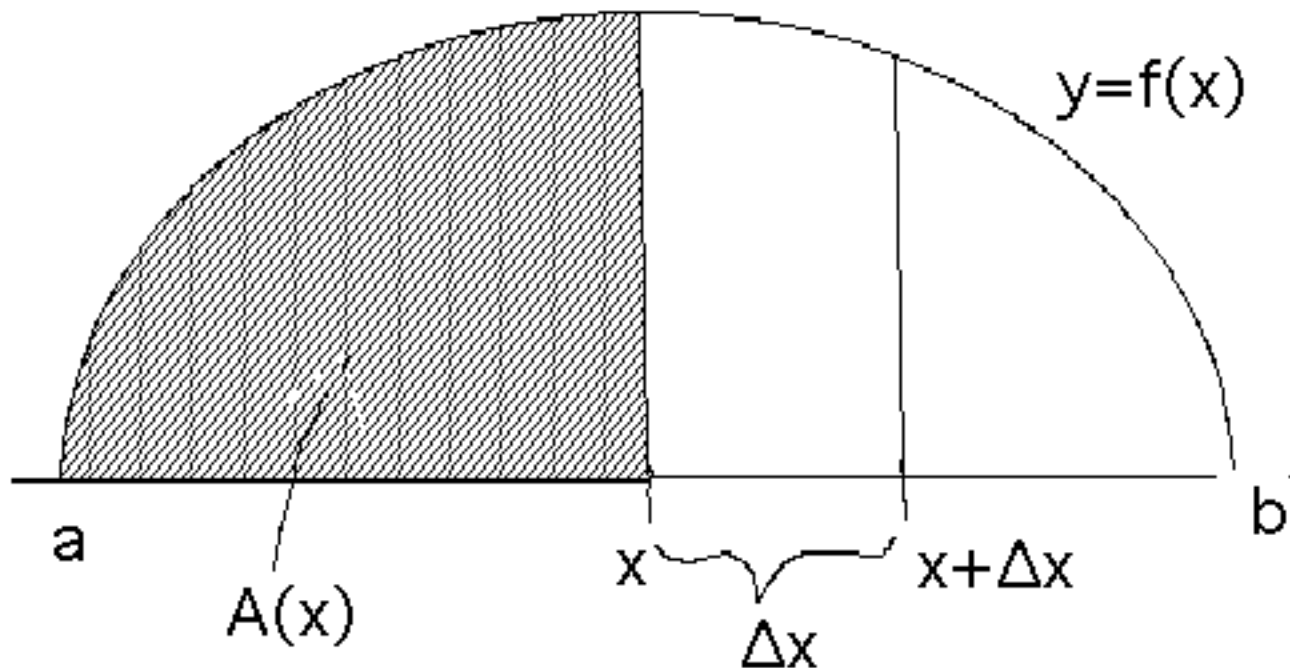
First remember how to calculate a derivative. If g is a function of x ,

the derivative of g at x , written dg/dx , is:

the limit of $\Delta g / \Delta x$, as $\Delta x \rightarrow 0$,

where Δx = "change in x ", and
 $\Delta g = g(x+\Delta x) - g(x)$ = "change in g ".

Let's review why the derivative of the area function for a graph is the height function, at least for continuous height functions.

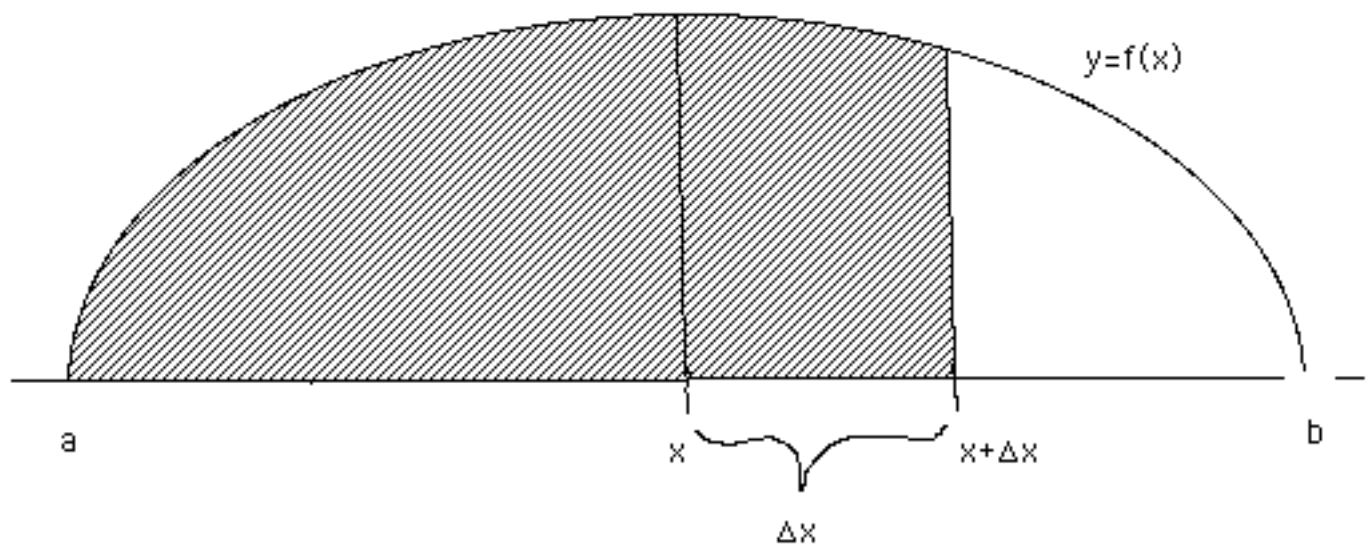


Define a function $A(x)$ = "that part of the area under the graph of a continuous f , measured from a to x ."

Then we claim the derivative of A at x , is simply $f(x)$, the height of the graph, at x .

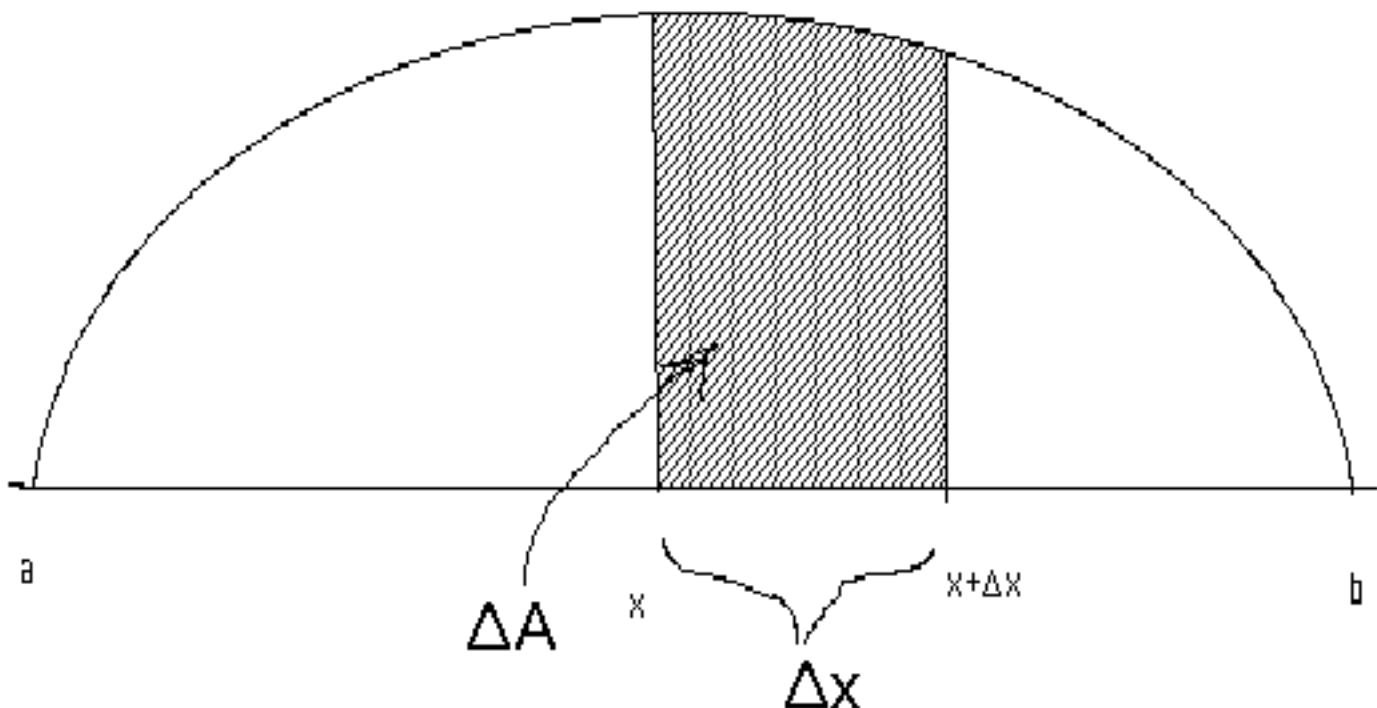
We must evaluate $\Delta A / \Delta x$, and take the limit as $\Delta x \rightarrow 0$. We see from the picture above that Δx = "change in x ", is the (length of the) interval of the x axis between x and $x+\Delta x$.

We know also that ΔA = "change in A " = $A(x+\Delta x) - A(x)$, so next we picture these. We have shaded the area $A(x)$ above, so next we shade $A(x+\Delta x)$ below.



$A(x+\Delta x)$ = shaded area = area between a and $x+\Delta x$.

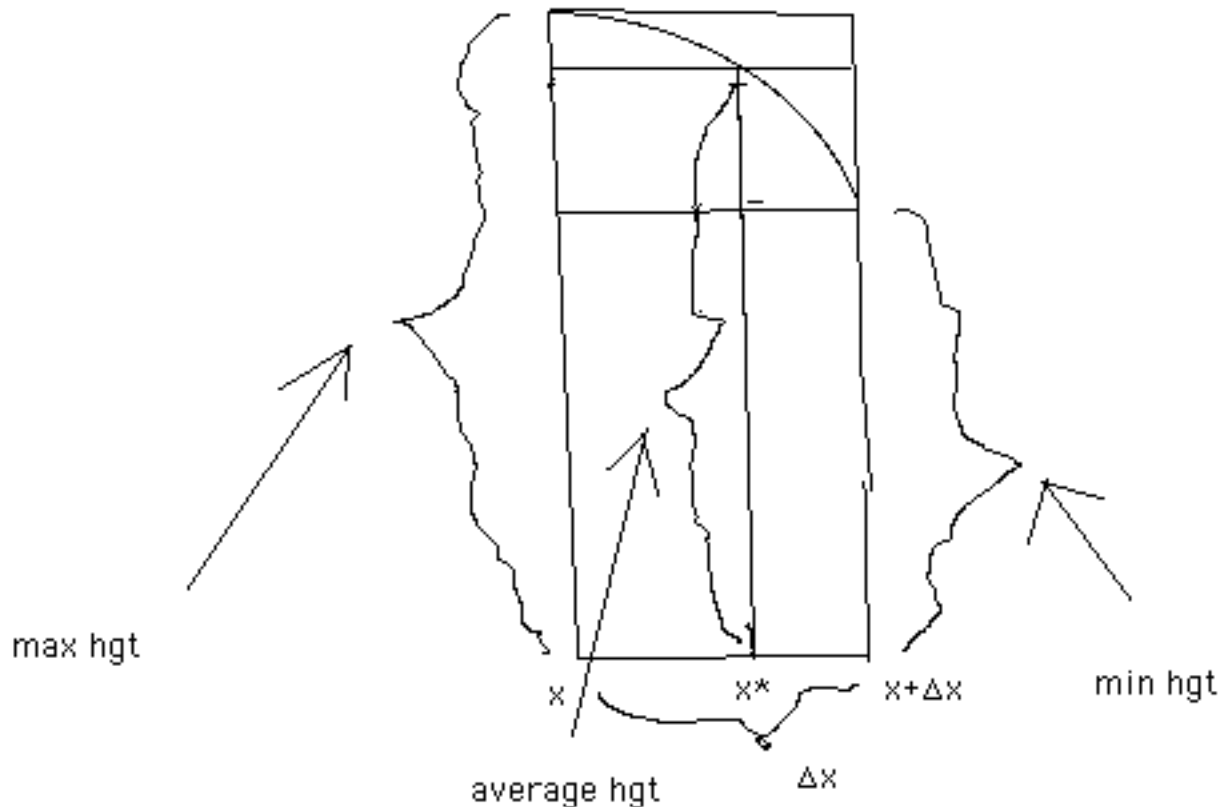
To get ΔA , we subtract, i.e. $\Delta A = A(x+\Delta x) - A(x)$ = shaded area below.



Now we have to divide $\Delta A/\Delta x$, and take the limit as $\Delta x \rightarrow 0$. I claim the quotient $\Delta A/\Delta x$, is the average height of the shaded region immediately above. I.e. when you multiply $\Delta A/\Delta x$ by Δx , you get the area ΔA of the shaded region. And Δx is the base of the shaded region. So ask

yourself what number, when multiplied by the base of the region, gives the area? The answer is just the average height, shown below.

At least, in our picture it is some number between the height at x and the height at $x+\Delta x$.



Now since $\Delta A/\Delta x$ is the average height of the graph between x and $x+\Delta x$, what is the derivative $dA/dx = \lim_{\Delta x \rightarrow 0} \Delta A/\Delta x$? For a continuous graph, this is just the actual height at x . I.e. for a continuous function, as the point x^* where the average height is taken, approaches x , the height at x^* approaches the height at x .

(Recall, f continuous at x means that, as x^* approaches x , the limit of $f(x^*)$ is $f(x)$.)

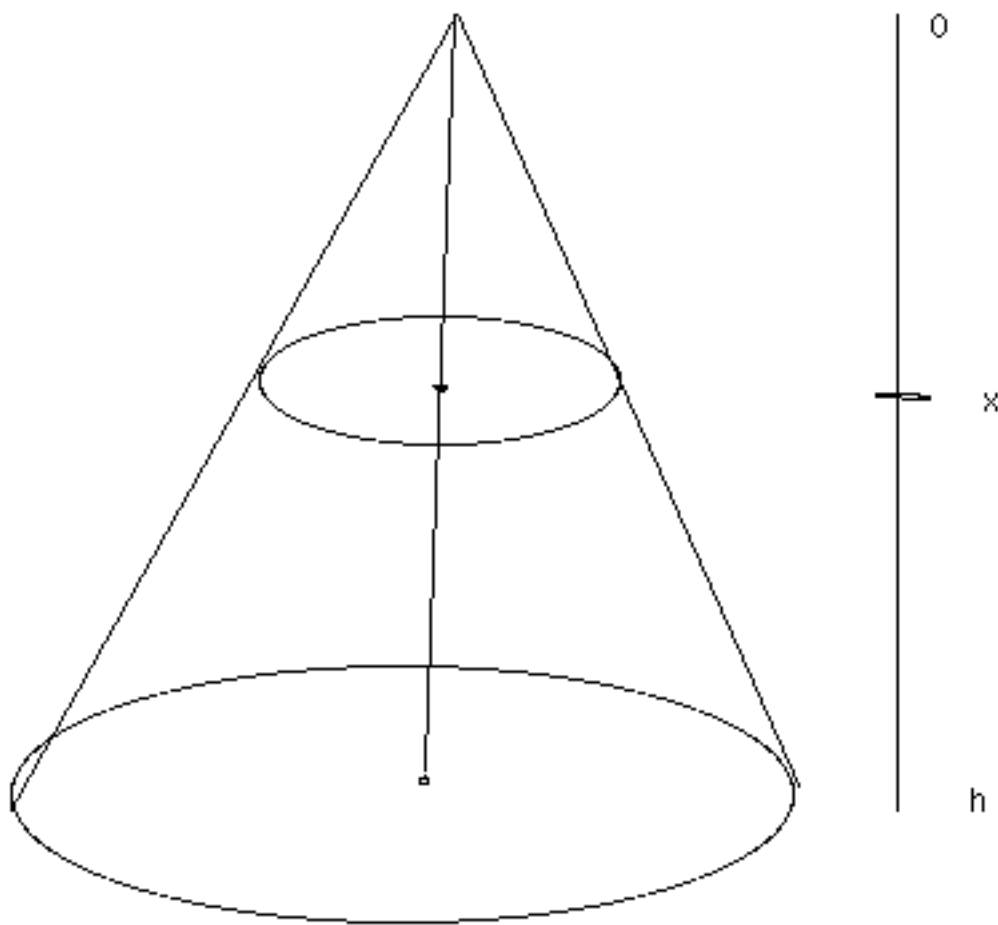
Application: If G is any antiderivative of $f(x)$, the area function for f is $G(x)-G(a)$.

proof: Two differentiable functions on an interval are equal if they have the same derivative, and are equal at one point. Now we just saw that $dA/dx = f(x)$, and by assumption also $d(G(x)-G(a))/dx = dG/dx = f(x)$ so we just need to compare $A(x)$ and $G(x)-G(a)$ at one point, such as a . But $A(a) = 0 = G(a)-G(a)$. Thus $A(x) = G(x)-G(a)$. I.e. if we can guess any antiderivative of f , we can guess the area function for f .

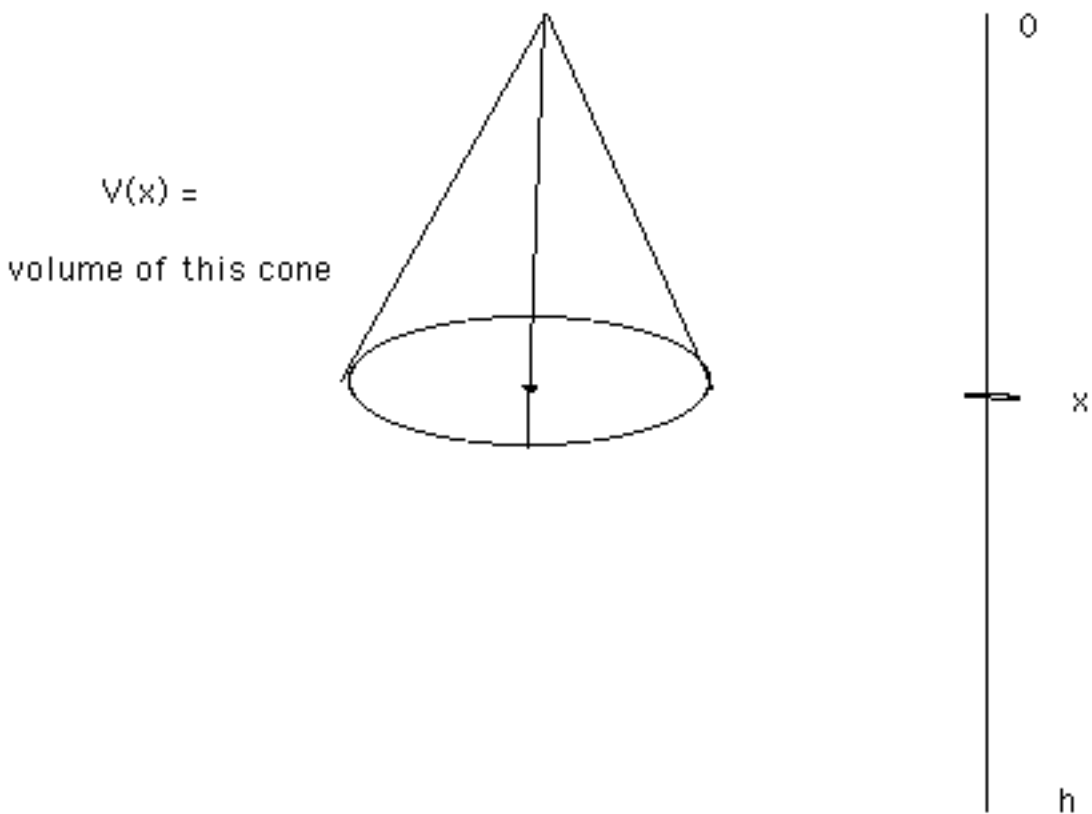
Volume of a cone.

Next we try to find the volume formula of a cone, by the same method. I.e. first try to find the

derivative of the volume function. Look at the picture below of a cone.



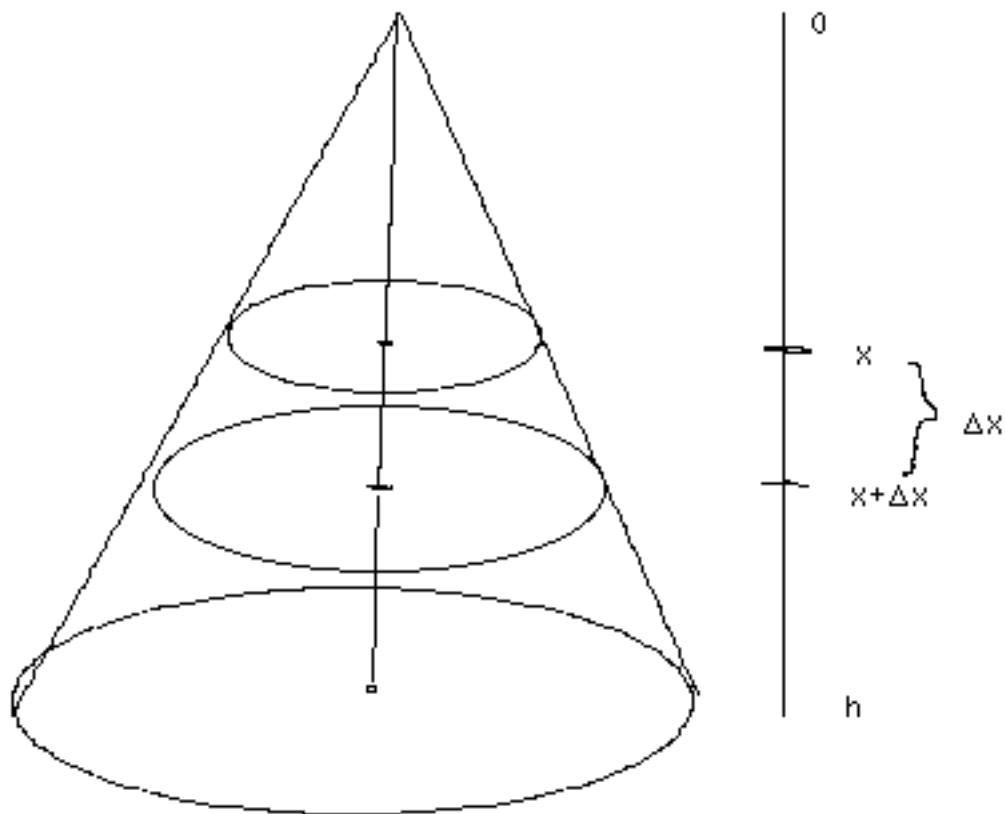
Now define a volume function V , where $V(x)$ = the part of the volume from the top of the cone down as far as distance x from the top, i.e. $V(x)$ is the volume of the small cone in the top part of the picture above. We show just this cone below.



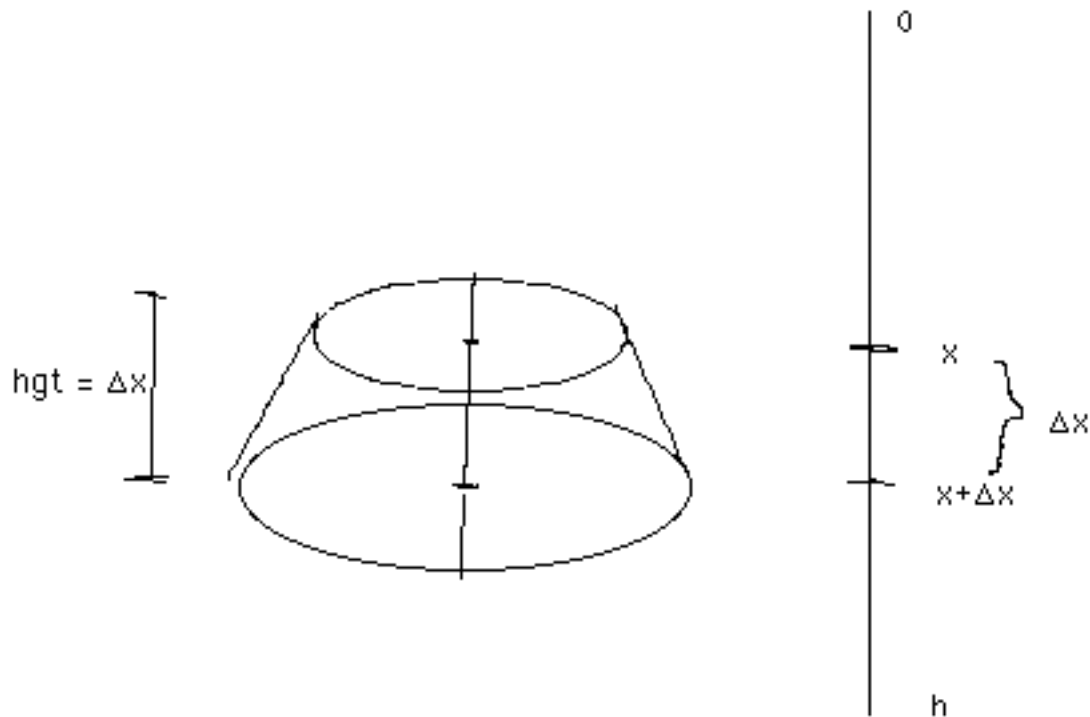
Then $V(h)$, where h is the height of the cone, is the volume of the big cone. To figure out what the formula for $V(x)$ is, we first ask what is the derivative dV/dx ?

To calculate it we again start from the definition, and compute ΔV and Δx .

In the next picture we have shown two values of x , x and $x+\Delta x$, and the two corresponding cones, one whose base is a distance x from the top, and the second one with base a distance $x+\Delta x$ from the top. $V(x)$ is the volume of the smaller cone, and $V(x+\Delta x)$ is the volume of the slightly larger one.



Then ΔV is the volume of the slab between the two cones, shown below.

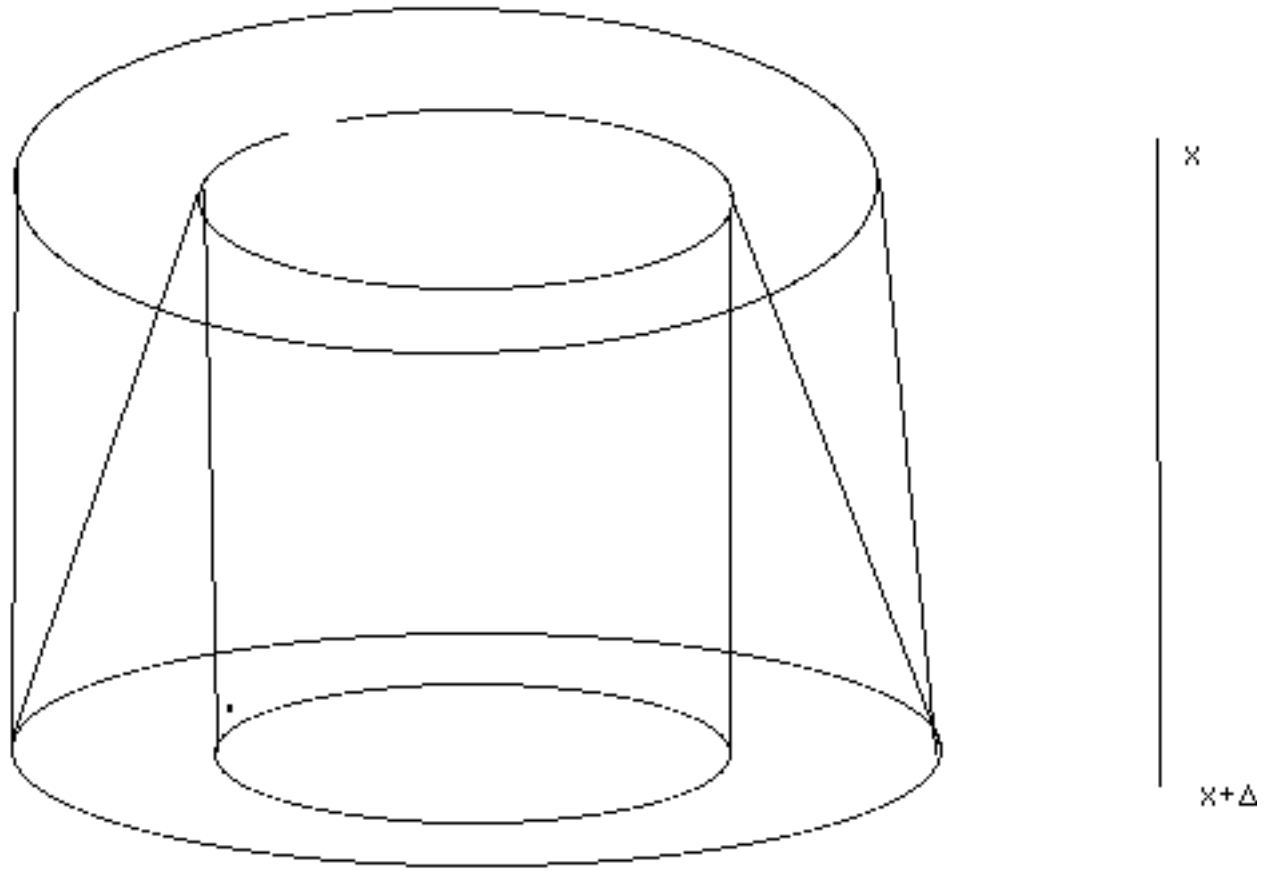


And we can see that Δx is the height of this slab.

Next we ask, what is the quotient $\Delta V / \Delta x$? Since $\Delta V = (\Delta V / \Delta x) \cdot \Delta x$, the quotient $(\Delta V / \Delta x)$ must be some number which gives the volume of the slab when you multiply it by the height of the slab.

I claim this is the area of some circle between the top circle and the bottom circle. I.e. I claim that if you multiply the height of the slab by the area of the bottom circle, you get more than the volume of the slab, while if you multiply the area of the top circle by the height of the slab, you get less volume than that of the slab.

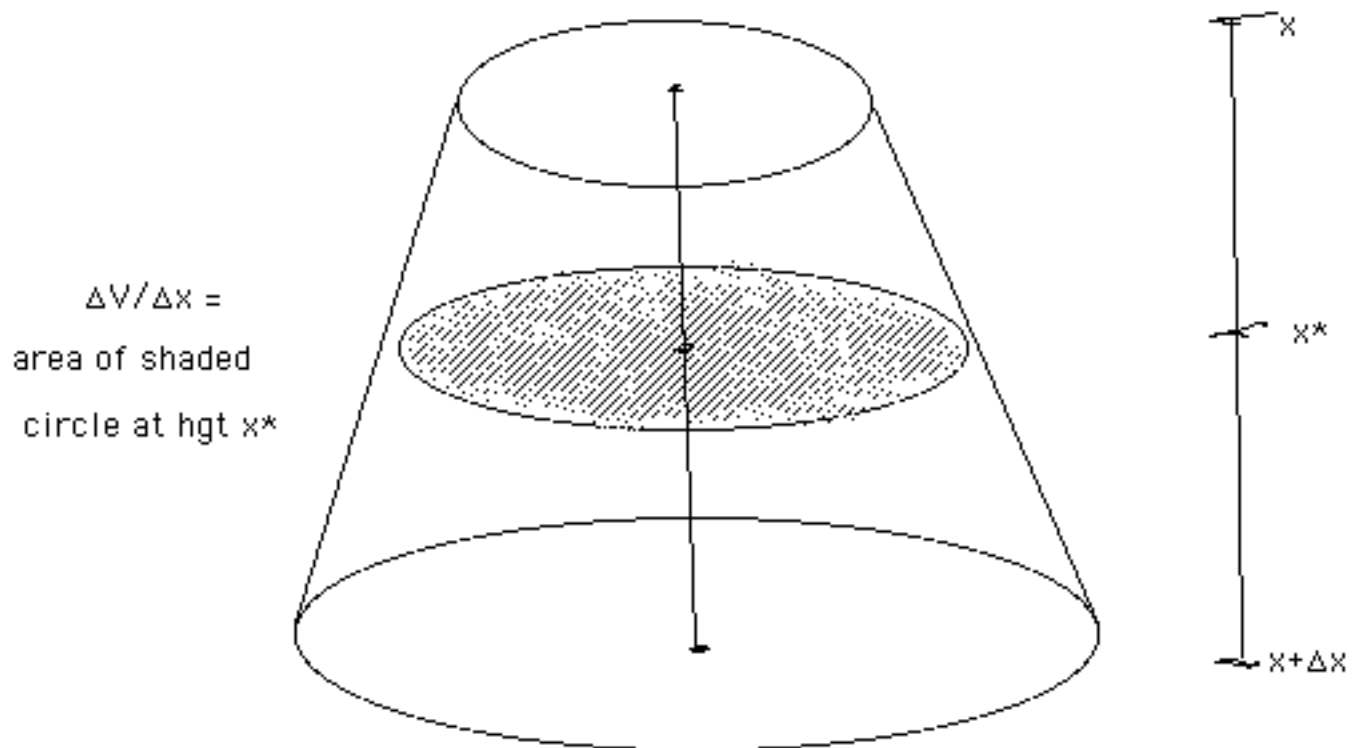
To see this, recall the volume of a cylinder, is obtained by multiplying the height by the area of the base. In our case the bottom circle is the base of a cylinder that is larger than the slab, and the top circle is the base of a cylinder smaller than the slab, but all having the same height.



Volume of big cylinder = Δx (area of large circle at bottom of conical slab)

Volume of small cylinder = Δx (area of small circle at top of conical slab) .

Thus volume of conical slab = Δx (area of some intermediate circle on conical slab).



Now to compute the derivative dV/dx , we ask what is the limit of $(\Delta V / \Delta x)$ as $\Delta x \rightarrow 0$? I.e. what is the limiting value of the shaded area, as $x + \Delta x \rightarrow x$?

The answer seems to be the area of the circle at height x .
I.e. $dV/dx =$ the area of the circle at height x .

I.e. the derivative of the volume function at x , seems to be the "area of a slice", at height x . So the derivative of a volume function seems to be an area function.

Lets apply this to find the volume of a cone and of a sphere. By our reasoning, we just need to find the corresponding area formulas, and then integrate.

Going back to the original cone, we ask for the area formula of the circle shown at distance x from the top. Of course we just need its radius. If the radius of the bottom circle is R , then by similar triangles we have $x/h = r/R$, so $r = xR/h$. Thus the area function $A(x) = \pi r^2 = \pi(xR/h)^2/h^2 = (\pi R^2/h^2)x^2$. Thus an antiderivative is

$$G(x) = (\pi R^2/h^2)x^3/3.$$

Since the top has $x = 0$, the volume function is

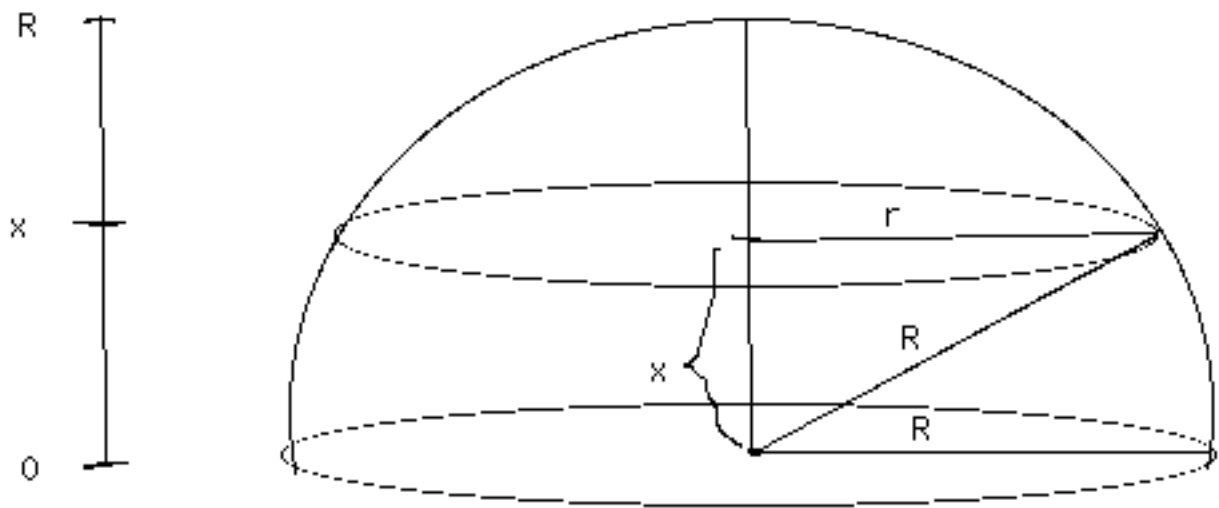
$$V(x) = G(x) - G(0) = G(x) = (\pi R^2/h^2)x^3/3.$$

Thus the full volume is $V(h) = (\pi R^2/h^2)h^3/3 = \pi R^2 h/3$.

This is $1/3$ the area of the base times the height, as expected.

Next we do the case of a hemisphere of radius R .

look at the picture below. If we define a volume formula $V(x)$ to equal the part of the volume of the hemisphere up only as far as height x , then the derivative of this function, by reasoning exactly similar to that above, is the area of a slice, again circular, at height x . So we need a formula for that area, and hence for the radius of that circle. Looking at the picture suggests using Pythagoras to get it.



I.e. by Pythagoras, we have $x^2 + r^2 = R^2$, so $r^2 = R^2 - x^2$. Thus the area we want is $\pi r^2 = \pi(R^2 - x^2)$. Thus $dV/dx = \pi(R^2 - x^2) = \pi R^2 - \pi x^2$, so an antiderivative is $G(x) = \pi R^2 x - \pi x^3/3$, and $G(0) = 0$, so again $V(x) = G(x) = \pi R^2 x - \pi x^3/3$. Thus the volume of the hemisphere is $V(R) = \pi R^3 - \pi R^3/3 = (2/3)\pi R^3$. Thus the volume of the full sphere is double this, or $(4/3)\pi R^3$, as discovered by Archimedes.

Exercise: Let a solid be formed by revolving the graph of $y = e^x$ around the x axis, between $x=0$ and $x=4$. Define a volume function $V(x)$ = the part of the volume inside this solid, between 0 and x .

Draw the picture of this solid.

What is the derivative of the volume function dV/dx ?

Find the volume of the solid.

Exercise: Let a solid be formed with base on the plane $z = 1$, one unit above the x,y plane, and shaped something like the Eiffel tower, with each cross section (slice) parallel to the x,y plane and at height z , being a square of side $1/z$. if the tower reaches from $z=1$ up to $z = 12$, find the volume of the tower.

(Draw the picture. Define a volume function to be $V(z)$ = that part of the volume between height 1 and height z . Find the derivative of this volume function. Find an antiderivative if possible, and find the volume.)