2310H Test 2 Fall 2004, NAME:
no calculators, good luck! (use the backs)

1. (a) Give the definition of "Lipschitz continuity" for a function f on an interval I.
(b) State a criterion for recognizing Lipschitz continuity in the case of a differentiable function f on an interval I.
(c) Determine which of the following functions is or is not Lipschitz continuous, and explain briefly why in each case.
(i) The function is $f(x) = x^{(1/3)}$, on the interval $(0, \infty)$.
(ii) The function is $G(x)$ = integral over $[0,x]$ of $f(t) = [t]$, for all x in the interval $[0,10]$. (Recall $[t]$ = "the greatest integer not greater than t": $[t] = 0$ for $0 \le t<1$, $[t] = 1$ for $1 \le t < 2$, $[t] = 2$ for $2 \le t < 3$, etc...., $[t] = 9$ for $9 \le t < 10$, $[10] = 10$.)
(iii) The function is $h(x) = x + \cos(x)$ on the interval $(-\infty,\infty)$.

2. (i) State the "fundamental theorem of calculus" (part 1), i.e. state the key properties of the indefinite integral function $G(x)$ = integral of f on the interval $[a,x]$, associated to an integrable function f on a closed bounded interval $[a,b]$. You may assume f is continuous everywhere on $[a,b]$ if you wish.

(ii) Explain carefully why the definite integral over the interval $[a,b]$ of a continuous function f, equals $H(b)-H(a)$, whenever H is any "antiderivative" of f, i.e. whenever $H'(x) = f(x)$ for all x in $[a,b]$. Justify the use of any theorems to which you appeal by verifying their hypotheses. (You are deriving part 2 of the FTC from part 1.)

(iii) Is there a differentiable function $G(x)$ with $G'(x) = \cos(x^2)$?
If so, give one, if not say why not.

3. Let S be the solid obtained by revolving the graph of $y = e^x$ around the x axis between $x=0$ and $x=3$. Define the moving volume function $V(x)$ = that part of the volume of S lying between 0 and x. (draw a picture.)

(i) What is $dV/dx = ?$

(ii) Write an integral for the volume of S, and compute that volume.

4. Consider a pyramid of height H, with base a square of side B. Define a moving volume function $V(x)$ = that part of the volume of the pyramid lying between the top of the pyramid, and a plane which is parallel to the base and at a distance x from the top.
(i) Find the derivative dV/dx. [Hint: Use similarity.]
(ii) Find the volume $V(H)$.
(iii) Make a conjecture about the volume of a pyramid of height H with base of any planar shape whatsoever, and base area B.

EXTRA: Prove the FTC (part 1), from part 2(i), you may draw pictures and assume your f is monotone and continuous if you like.