6.6. Physical Applications.

14. Find the mass of a thin bar with density function

$$\rho(x) = \begin{cases} 1 & : 0 \leq x \leq 2 \\ 2 & : 2 < x \leq 3 \end{cases}$$

Solution: We will need two integrals

$$\begin{aligned} m &= \int_0^3 \rho(x)\, dx \\ &= \int_0^2 \rho(x)\, dx + \int_2^3 \rho(x)\, dx \\ &= \int_0^2 1\, dx + \int_2^3 2\, dx \\ &= [x]_0^2 + [2x]_2^3 \\ &= (2-0) + (6-4) \\ &= 4 \end{aligned}$$

21. It takes 100 J of work to stretch a spring 0.5 m from its equilibrium position. How much work is needed to stretch it an additional 0.75 m?

Solution: Recall from Hooke's law that the force required to maintain the spring in a stretched position is $F(x) = kx$, where k is a constant. Thus,

$$\begin{aligned} 100 &= \int_0^{0.5} F(x)\, dx \\ &= \int_0^{0.5} kx \\ &= \left[\frac{k}{2}x^2\right]_0^{0.5} \\ &= \frac{k}{2}\left[(0.5)^2 - (0)^2\right] \\ &= \frac{k}{8} \end{aligned}$$

So $k = 800$. Then, the work needed to stretch the spring an additional 0.75 m is

$$\begin{aligned} W &= \int_{0.5}^{1.25} 800x\, dx \\ &= [400x^2]_{0.5}^{1.25} \\ &= [400(1.25)^2 - 400(0.5)^2] \\ &= 525 \text{ J} \end{aligned}$$

24. A cylindrical water tank has height 8 m and radius 2 m.

(a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?
(b) Is it true that it takes half as much work to pump the water out of the tank when it is half full as when it is full?

Solution: We place a coordinate axis with its origin at the top of the tank and positive values increasing downwards towards the bottom of the tank.

(a) We partition the interval $[0, 8]$ into equal subintervals, and consider this as partitioning the tank into slices of water. We consider the ith slice of water of thickness Δx m located x_i m from the top of the tank. This slice has volume $\pi(2)^2\Delta x = 4\pi\Delta x$ m^3. Since the density of water is 1000 kg/m^3, the slice has mass $4000\pi\Delta x$ kg. The force needed to move the slice must overcome gravity, so this force is $F_i = (4000\pi\Delta x)(9.8) = 39200\pi\Delta x$ N. The slice must move x_i m to the top of the tank, so the work needed to move the slice is $W_i = F_i x_i = 39200\pi x_i\Delta x$ J. By summing up the work needed to move all slices, we obtain an approximation of the work required to empty the tank. By taking the limit as the number of subintervals goes to infinity, we obtain the actual work needed to empty the tank. Thus

$$
\begin{aligned}
W &= \lim_{n\to\infty}\sum_{i=1}^{n} W_i \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 39200\pi x_i\Delta x \\
&= \int_0^8 39200\pi x\,dx \\
&= 19600\pi x^2 \\
&= 19600\pi(8)^2 - 19600\pi(0)^2 \\
&= 1,254,400\pi \text{ J}
\end{aligned}
$$

(b) Retaining the notation and preliminary calculations from the previous part, when the tank is half full, the water is located from $x = 4$ to $x = 8$. Thus, the work needed to pump the water out of the tank when it is half full is

$$
\begin{aligned}
W &= \int_4^8 39200\pi x\,dx \\
&= 19600\pi x^2 \\
&= 19600\pi(8)^2 - 19600\pi(4)^2 \\
&= 940,800\pi \text{ J}
\end{aligned}
$$

which is not half of the value found in part (a).

34. The lower edge of a dam is defined by the parabola $y = x^2/16$. Use a coordinate system with $y = 0$ at the bottom of the dam to determine the total force on the dam. Lengths are measured in meters.

Solution: As suggested, we place the origin of the usual coordinate axes at the bottom of the dam. We partition the interval $[0, 25]$ into equal subintervals, and consider this as slicing the dam into slices of equal width. We consider the ith slice of the dam of width Δy m at a height of y_i m from the bottom. Solving $y = x^2/16$ for x in terms of y gives $x = \pm\sqrt{16y} = \pm 4\sqrt{y}$. The width of the slice is thus $(4\sqrt{y_i}) - (-4\sqrt{y_i}) = 8\sqrt{y_i}$ m. The strip has area $8\sqrt{y_i}\,\Delta y$ m^2. The density of water is $\rho = 1000$ kg/m^3, and the acceleration due to gravity is $g = 9.8$ m/s^2. Thus, the total force on the ith slice of the dam is approximately $F_i = 1000(9.8)(25 - y_i)(8\sqrt{y_i})\Delta y$. By summing up the forces on each strip of the dam, we obtain an approximation of the total force on the dam. Letting the number of slices go to infinity gives us the total force on the dam:

$$
\begin{aligned}
F &= \lim_{n\to\infty} \sum_{i=1}^{n} F_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 1000(9.8)(25 - y_i)(8\sqrt{y_i})\Delta y \\
&= \int_0^{25} 1000(9.8)(25 - y)(8\sqrt{y})\,dy \\
&= 78400 \int_0^{25} 25\sqrt{y} - y^{3/2}\,dy \\
&= 78400 \left[\frac{50}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^{25} \\
&= 78400 \left[\left(\frac{50}{3}(25)^{3/2} - \frac{2}{5}(25)^{5/2}\right) - \left(\frac{50}{3}(0)^{3/2} - \frac{2}{5}(0)^{5/2}\right)\right] \\
&= 78400 \left[\frac{2500}{3} - 0\right] \\
&= \frac{196000000}{3}\ \text{J}
\end{aligned}
$$