Prove that the current $\mathbf{J}$ *can be decomposed into longitudinal and transverse components,*

$$\mathbf{J} = \mathbf{J}_\ell + \mathbf{J}_t \qquad \text{[I.1]}$$

Helmholtz theorem, for a bit of justification.

$$\forall \mathbf{F} \exists U, \vec{\mathbf{W}} : \vec{\mathbf{F}} \equiv -\vec{\nabla} U + \vec{\nabla} \times \vec{\mathbf{W}} \qquad \text{[I.2]}$$

We see that $\nabla \times (\nabla U) = \mathbf{0}$ and $\nabla \bullet (\nabla \times \mathbf{W}) \equiv 0$; essentially, we are breaking up any ol' vector function into its divergenceless component and its curlless component.

Let,

$$\mathbf{J}_t \equiv \frac{1}{4\pi} \nabla \times \left(\nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \right) \qquad \text{[I.3]}$$

$$\mathbf{J}_\ell \equiv -\frac{1}{4\pi} \nabla \int \frac{\nabla' \bullet \mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \overset{????}{=} -\frac{1}{4\pi} \nabla \left(\nabla \bullet \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \right) \qquad \text{[I.4]}$$

Does $\nabla' \to \nabla$ if we bring it outside the integral? Well, I can't go any further if I don't use this; It can be shown that [I.3] and [I.4] tally to $\mathbf{J}$ in agreement with [I.2]. Add them and combine the vector-derivatives under the integral sign,

$$\mathbf{J}_\ell + \mathbf{J}_t = -\frac{1}{4\pi} \vec{\nabla} \left(\vec{\nabla} \bullet \int \frac{\mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \right) + \vec{\nabla} \times \left(\frac{1}{4\pi} \vec{\nabla} \times \int \frac{\mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \right) = \frac{1}{4\pi} \int \frac{\vec{\nabla} \times (\vec{\nabla} \times \mathbf{J}) - \vec{\nabla}(\vec{\nabla} \bullet \mathbf{J})}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \qquad \text{[I.5]}$$

We need to realize $\nabla \times (\nabla \times \mathbf{A}) \equiv \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$, and [I.5] becomes,

$$\mathbf{J}_\ell + \mathbf{J}_t = \frac{1}{4\pi} \int \frac{\nabla(\nabla \bullet \mathbf{J}) - \nabla^2 \mathbf{J} - \nabla(\nabla \bullet \mathbf{J})}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' = -\frac{1}{4\pi} \int \frac{\nabla^2 \mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \qquad \text{[I.6]}$$

Integrating by parts using $\nabla(\nabla \bullet (\mathbf{A}\psi)) \equiv \nabla(\mathbf{A} \bullet \nabla \psi + \psi \cdot \nabla \bullet \mathbf{a}) = \left(\nabla(\mathbf{A} \bullet \nabla \psi) \right) + \nabla(\psi \cdot \nabla \bullet \mathbf{a})$

Thing we want:

$$\frac{-1}{4\pi} \int \frac{\nabla^2 \mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \overset{???}{=} \frac{-1}{4\pi} \int \left(\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathbf{J}(\mathbf{x}') d^3 \mathbf{x}' = \int \frac{-1}{4\pi} \left(\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathbf{J}(\mathbf{x}') d^3 \mathbf{x}' = \int \delta(\mathbf{x} - \mathbf{x}') \mathbf{J}(\mathbf{x}') d^3 \mathbf{x}' = \mathbf{J}(\mathbf{x}) \qquad \text{[I.7]}$$

We see that [I.7] implies,

$$\int \frac{\nabla^2 \mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \overset{???}{=} \int \left(\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathbf{J}(\mathbf{x}') d^3 \mathbf{x}' \qquad \text{[I.8]}$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \nabla^2 \mathbf{J} = \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial^2 J_i}{\partial x_i^2} \right) \hat{e}_i \qquad \text{[I.9]}$$

Second derivative of product of scalar functions,

$$\frac{\partial^2}{\partial x_i^{\,2}}\left(\frac{J_i}{|\mathbf{x}-\mathbf{x}'|}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial J_i}{\partial x_i}\frac{1}{|\mathbf{x}-\mathbf{x}'|}+J_i\frac{\partial}{\partial x_i}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\frac{\partial^2 J_i}{\partial x_i^{\,2}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}+2\frac{\partial J_i}{\partial x_i}\frac{\partial}{\partial x_i}\frac{1}{|\mathbf{x}-\mathbf{x}'|}+J_i\frac{\partial^2}{\partial x_i^{\,2}}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \qquad \text{[I.10]}$$

Clearly, then,

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\nabla^2\mathbf{J}&=\left(\frac{\partial^2}{\partial x_i^{\,2}}\left(\frac{J_i}{|\mathbf{x}-\mathbf{x}'|}\right)-2\frac{\partial J_i}{\partial x_i}\frac{\partial}{\partial x_i}\frac{1}{|\mathbf{x}-\mathbf{x}'|}-J_i\frac{\partial^2}{\partial x_i^{\,2}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\hat{e}_i\\
&=\nabla^2\left(\frac{J_i}{|\mathbf{x}-\mathbf{x}'|}\right)-2(\nabla\bullet\mathbf{J})\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)-\mathbf{J}(-4\pi\cdot\delta(\mathbf{x}-\mathbf{x}'))\\
&=\nabla^2\left(\frac{J_i}{|\mathbf{x}-\mathbf{x}'|}\right)-2(\nabla\bullet\mathbf{J})\left(\frac{-\frac{1}{2}2(x_i-x_i')(1-0)}{|\mathbf{x}-\mathbf{x}'|^3}\hat{e}_i\right)-\mathbf{J}(-4\pi\cdot\delta(\mathbf{x}-\mathbf{x}'))\\
\frac{1}{|\mathbf{x}-\mathbf{x}'|}\nabla^2\mathbf{J}&=\nabla^2\left(\frac{J_i}{|\mathbf{x}-\mathbf{x}'|}\right)+2(\nabla\bullet\mathbf{J})\frac{x_i-x_i'}{|\mathbf{x}-\mathbf{x}'|^3}\hat{e}_i+\mathbf{J}\cdot 4\pi\cdot\delta(\mathbf{x}-\mathbf{x}')\end{aligned} \qquad \text{[I.11]}$$

The vector laplacian is expressible as the gradient of a divergence,

$$\nabla^2(\mathbf{a}\psi)=\nabla\left(\nabla\bullet(\mathbf{a}\psi)\right)=\nabla\left(\mathbf{a}\bullet\nabla\psi+\psi\nabla\bullet\mathbf{a}\right)=\nabla(\mathbf{a}\bullet\mathbf{v})+\nabla(\psi\nabla\bullet\mathbf{a})=\begin{pmatrix}(\mathbf{a}\bullet\nabla)\mathbf{v}+(\mathbf{v}\bullet\nabla)\mathbf{a}\\+\mathbf{a}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{a})\end{pmatrix}+\begin{pmatrix}(\nabla\psi)(\nabla\bullet\mathbf{a})+\\\psi(\nabla^2\mathbf{a})\end{pmatrix} \qquad \text{[I.12]}$$

Restoring our abbreviation $\mathbf{v}=\nabla\psi$, we see [I.12] becomes the identity,

$$\begin{aligned}\nabla^2(\mathbf{a}\psi)&=\left((\mathbf{a}\bullet\nabla)(\nabla\psi)+((\nabla\psi)\bullet\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times(\nabla\psi))+(\nabla\psi)\times(\nabla\times\mathbf{a})\right)+(\nabla\psi)(\nabla\bullet\mathbf{a})+\psi(\nabla^2\mathbf{a})\\
&=(\mathbf{a}\bullet\nabla)(\nabla\psi)+((\nabla\psi)\bullet\nabla)\mathbf{a}+(\nabla\psi)\times(\nabla\times\mathbf{a})+(\nabla\psi)(\nabla\bullet\mathbf{a})+\psi(\nabla^2\mathbf{a})\end{aligned} \qquad \text{[I.13]}$$

fObviously, $\mathbf{a}\rightarrow\mathbf{J}$ and $\psi\rightarrow\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, so the LHS of [I.8] becomes,

$$\nabla^2(\mathbf{a}\psi)=\left((\mathbf{a}\bullet\nabla)\mathbf{v}+(\mathbf{v}\bullet\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{a})\right)+(\nabla\psi)(\nabla\bullet\mathbf{a})+\psi(\nabla^2\mathbf{a}) \qquad \text{[I.14]}$$