I've tried to give elementary proofs of the "three big theorems" on continuity used in elementary calculus (taken together they say the continuous image of a closed bounded interval is again a closed bounded interval). I suggest this can be presented at least in a typical first year honors calculus class. First they should know the epsilon – delta definition of continuity.

1) Intermediate value theorem: Assume f continuous on $[0,1]$, and assume $f(0)<0<f(1)$. Then by looking at the values $f(0)$, $f(0.1)$, $f(0.2)$,…,$f(.9)$, $f(1.0)$, there is some integer a1 between 0 and 9 so that $f(.a1) \leq 0 \leq f(.a1+.1)$. If one of these values is zero we stop.
If not, then there is some integer a2 so that $f(.a1a2) \leq 0 \leq f(.a1a2 +.01)$.
Continue…..
Either we find a point where $f = 0$ or else we find a sequence of decimals $xn = .a1a2….an$, and $xn+1 = = .a1a2….an + 1/10^n$, so that $f(xn) < 0 < f(xn+1)$ for all n, and $|xn - xn+1| < 1/10^n$.

Since both sequences $\{xn\}$ and $\{xn + 1/10^n\}$ converge to the same decimal $x = a1a2a3…….$, and since all $f(xn)<0$ while all $f(xn +1/10^n) >0$, it follows that $0 \leq f(x) \leq 0$. QED.

Here are two more such arguments along the same lines.

2) Every function f continuous on $[0,1]$ is bounded there.
proof: if not then it is unbounded on some interval of form $[.a1, .a1+.1]$,
hence also on some interval of form $[.a1a2, .a1a2 +.01]$.

Continuing we find an infinite decimal $x = .a1a2a3....$ in $[0,1]$, such that f is unbounded on every interval containing x. But if f is continuous at x, then f is bounded on some neighborhood of x. QED.

3) Claim: A continuous f takes on a maximum on $[0,1]$.
proof: By theorem 1 above (IVT) the set of values of f form an interval, and by theorem 2), they form a bounded interval. If that interval is not closed on the right it has form say (c,d), but then the continuous function $1/(f-d)$ would be unbounded on $[0,1]$. QED.

My hope is to be more consistent and logical for the average calculus students trying to follow the ideas. I.e. if we tell them in a non honors class, or even an honors class that real numbers are infinite decimals, why not use that statement to give them direct proofs of the big theorems, instead of just saying "this is beyond the scope of the course", when really it isn't at all. I always loved abstraction, but I now believe after a lifetime of teaching that unnecessarily abstract presentations cause many students to just lose contact with the subject.

What do you think? (Although I suggest this as an alternative to the usual axiomatic approach found in Spivak and elsewhere, I learned the rigorous approach to real numbers via decimals from an appendix in Spivak, while teaching a group of bright high schoolers, so in a way Mike also deserves credit for this approach.)

I also recommend using words in discussing these theorems. I.e. many students who cannot regurgigate correctly that for a continuous f on $[a,b]$ with $f(a) < 0 < f(b)$, there exists c with $a < c < b$ and $f(c) = 0$, can still say correctly “the continuous image of an interval is also an interval”.