4. Linear Transformations

DEFINITION 47. *Let V and U be vector spaces. A linear transformation from V to U is a map*

$$T : V \to U$$

with the following two properties for all $\mathbf{v}, \mathbf{w} \in V$ and all scalars r :

$$\begin{aligned} T(r\mathbf{v}) &= rT(\mathbf{v}) \\ T(\mathbf{v}+\mathbf{w}) &= T(\mathbf{v}) + T(\mathbf{w}) \end{aligned}$$

EXAMPLE 72. *As before, if $V = \Re^n$ and $U = \Re^m$, and if M is a matrix with n columns and m rows, then $\vec{v} \mapsto M\vec{v}$ is linear.*

EXAMPLE 73. *If $V = C[0,1]$, the vector space of continuous functions on $[0,1]$, then the map $I : C[0,1] \to \Re$ defined by*

$$I(f) = \int_0^1 f(t)\,dt$$

is linear.

$$\begin{aligned} I(rf) &= \int_0^1 (rf(t))\,dt \\ &= r\int_0^1 f(t)\,dt \\ &= rI(f) \end{aligned}$$

$$\begin{aligned} I(f+g) &= \int_0^1 (f(t)+g(t))\,dt \\ &= \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt \\ &= I(f) + I(g) \end{aligned}$$

EXAMPLE 74. *If $V = C[0,1]$, and if $r \in [0,1]$, then the map $\varepsilon_r : C[0,1] \to \Re$ defined by*

$$\varepsilon_r(f) = f(r)$$

is linear.

$$\begin{aligned} \varepsilon_r(sf) &= (sf)(r) \\ &= sf(r) \\ &= s\varepsilon_r(f) \end{aligned}$$

$$\begin{aligned} \varepsilon_r(f+g) &= (f+g)(r) \\ &= f(r) + g(r) \\ &= \varepsilon_r(f) + \varepsilon_r(g) \end{aligned}$$

EXAMPLE 75. *If $V = C[0,1]$, and if $g \in C[0,1]$ is a fixed continuous map, then the map $T_g : C[0,1] \to C[0,1]$ defined by*

$$T_g(f)(x) = \int_0^x g(t)\,f(t)\,dt$$

is linear. - For example, we could take $g(x) = x^2$. Then

$$\begin{aligned} T_g(\sin\sigma)(x) &= \int_0^x g(t)\sin(t)\,dt = \int_0^x t^2\sin(t)\,dt \\ &= -x^2\cos x + 2\cos x + 2x\sin x - 2 \end{aligned}$$

- If $g(x) = x^3$. Then

$$\begin{aligned} T_g\left(1 - x + x^2\right)(x) &= \int_0^x t^3\left(1 - t + t^2\right)dt \\ &= \int_0^x \left(t^3 - t^4 + t^5\right)dt \\ &= \left[\frac{1}{4}t^4 - \frac{1}{5}t^5 + \frac{1}{6}t^6\right]_0^x \\ &= \frac{1}{4}x^4 - \frac{1}{5}x^5 + \frac{1}{6}x^6 \end{aligned}$$

This map is linear:

$$\begin{aligned} T_g(rf)(x) &= \int_0^x g(t)(rf(t))\,dt \\ &= \int_0^x rg(t)f(t)\,dt \\ &= r\int_0^x g(t)f(t)\,dt \\ &= rT_g(f)(x) \end{aligned}$$

$$\begin{aligned} T_g(f+h)(x) &= \int_0^x g(t)(f(t) + h(t))\,dt \\ &= \int_0^x [g(t)f(t) + g(t)h(t)]\,dt \\ &= \int_0^x g(t)f(t)\,dt + \int_0^x g(t)h(t)\,dt \\ &= T_g(f)(x) + T_g(h)(x) \end{aligned}$$

EXAMPLE 76. *If $V = C_1[0,1]$, the vector space of all continuously differentiable functions defined on $[0,1]$, then the map $d : V \to C[0,1]$ defined by*

$$d(f) = \frac{df}{dx}$$

is linear.

$$\begin{aligned} d(rf) &= \frac{d}{dx}(rf) \\ &= r\frac{df}{dx} \\ &= rdf \end{aligned}$$

$$\begin{aligned} d(f+g) &= \frac{d}{dx}(f+g) \\ &= \frac{df}{dx}+\frac{dg}{dx} \\ &= df+dg \end{aligned}$$

4.1. Properties of Linear Transformation.

Proposition 21. *If $T:V\to U$ is a linear transformation, then*
1. $T(\mathbf{o})=\mathbf{o}$*, and*
2. $T(-\mathbf{v})=-T(\mathbf{v})$.

Proof. We compute:

$$\begin{aligned} T(\mathbf{o}) &= T(\mathbf{o}+\mathbf{o}) \\ &= T(\mathbf{o})+T(\mathbf{o}) \text{ (2nd property)} \end{aligned}$$

So

$$T(\mathbf{o})=\mathbf{o} \text{ (subtract } T(\mathbf{o}) \text{ from both sides)}$$

And

$$\begin{aligned} T(-\mathbf{v}) &= T((-1)\mathbf{v}) \\ &= (-1)T(\mathbf{v}) \text{ (1st propery of linear maps)} \\ &= -T(\mathbf{v}) \end{aligned}$$

□

Proposition 22. *If $T:W\to V$ and $S:V\to U$ are linear transformation, then $S\circ T:W\to U$ is linear.*

Proof. We compute:

$$\begin{aligned} (S\circ T)(r\mathbf{v}) &= S(T(r\mathbf{v})) \text{ def. of composition} \\ &= S(rT(\mathbf{v})) \text{ 1st property of linearity applied to } T \\ &= rS(T(\mathbf{v})) \text{ 1st property of linearity applied to } S \\ &= r(S\circ T)(\mathbf{v}) \text{ def. of composition} \end{aligned}$$

and

$$\begin{aligned} (S\circ T)(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{u}+\mathbf{v})) \text{ def. of composition} \\ &= S(T(\mathbf{u})+T(\mathbf{v})) \text{ 2nd property of linearity applied to } T \\ &= S(T(\mathbf{u}))+S(T(\mathbf{v})) \text{ 2nd property of linearity applied to } S \\ &= (S\circ T)(\mathbf{u})+(S\circ T)(\mathbf{v}) \text{ def. of composition} \end{aligned}$$

□

Definition 48. *Let U and V be vector spaces. An isomorphism from U to V is a bijective linear transformation $T:U\to V$*

Bijective means: The map is one-to-one (injective), and onto (surjective).
One-to-one: If $T(x_1) = T(x_2)$ then $x_1 = x_2$.
Onto: For every given $\mathbf{v} \in V$ there is an element $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.
Bijective maps T have inverses T^{-1}

$$T(\mathbf{u}) = \mathbf{v} \iff \mathbf{u} = T^{-1}(\mathbf{v})$$

PROPOSITION 23. *If $T : U \to V$ is an isomorphism, then $T^{-1} : V \to U$ is also linear and therefore an isomorphism.*

PROOF. We have to show that

$$T^{-1}(r\mathbf{v}) = rT^{-1}(\mathbf{v})$$

Both are vectors in U, and

$$\begin{aligned} T\left(T^{-1}(r\mathbf{v})\right) &= r\mathbf{v} \\ T\left(rT^{-1}(\mathbf{v})\right) &= rTT^{-1}(\mathbf{v}) = r\mathbf{v} \end{aligned}$$

Since T is one-to-one, it follows that $T^{-1}(r\mathbf{v}) = rT^{-1}(\mathbf{v})$.
The same method gives the second property of linearity (see homework). □

THEOREM 58. *If U is a vector space with a finite basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, then $\mathbf{v} \mapsto [\mathbf{v}]_B : U \to \Re^n$ is an isomorphism. This inverse of this isomorphism is the linear transformation $c_B : \Re^n \to U$.*