PC3672

**ONE HOUR THIRTY MINUTES**

A list of constants is enclosed.

**UNIVERSITY OF MANCHESTER**

Mathematical Methods for Physics

30th May 2003, 2.00 p.m. - 3.30 p.m.

Answer **TWO** questions

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Electronic calculators may be used, provided that they cannot store text.

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The numbers are given as a guide to the relative weights of the different parts of each question.

**1.** Consider the Sturm-Liouville equation

$$L(x)\,y(x) = \lambda\rho(x)y(x) \qquad\qquad a < x < b$$

where $\lambda$ is the eigenvalue, $\rho(x) > 0$ is a real weight function and the Sturm-Liouville operator is

$$L(x) = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x)$$

where $p(x),\ q(x)$ are given real functions.

Show that for any two functions $u(x)$ and $v(x)$,

$$vLu - uLv = (p\,u\,v' - p\,v\,u')'\ .$$

[3 marks]

Hence derive a general criterion which the boundary conditions on the solutions must satisfy at $x = a, b$ for $L(x)$ to be an Hermitian operator. Give an example of boundary conditions which satisfy this criterion for the cases:

(a) $p(x) = 0$ at $x = a$ and $x = b$; and

(b) $p(x) \neq 0$ at $x = a$ and $x = b$.

[9 marks]

The Legendre equation is

$$\frac{d}{dx}\,(1-x^2)\frac{dy}{dx} + \ell\,(\ell+1)y = 0$$

where $\ell$ is an integer. Given a solution $u(x)$, obtain an expression for a second solution $v(x)$ corresponding to the same value of $\ell$.

[9 marks]

For $\ell = 1$ one easily verifies that the Legendre equation is satisfied by $u(x) = x$. Find the corresponding second solution $v(x)$, evaluating explicitly any integrals which may occur.

[4 marks]

**2.** Suppose that $G(\mathbf{x}, t; \mathbf{x}', t')$ is the Green function for the wave equation. Why is the boundary condition usually chosen to be $G(\mathbf{x}, t; \mathbf{x}', t') = 0$ for $t < t'$? Why is it sufficient to only know the Green function for $x' = 0,\ t' = 0$?

[4 marks]

The following expression for the Green function for the wave equation in one space and one time dimension can be derived using Fourier transforms :

$$G(x, t; 0, 0) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{c^2}{\omega^2 - k^2 c^2} e^{i(kx-\omega t)}.$$

where the symbols have their usual meaning.

Sketch a suitable contour for doing the $\omega$ integral for $t < 0$, assuming $G(x, t; 0, 0) = 0$ in this case. Draw clearly the path that must be chosen around the poles on the real axis.

[6 marks]

Show that for $t > 0$,

$$G(x, t; 0, 0) = \frac{iA}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k} \left[ e^{ik(x-ct)} - e^{ik(x+ct)} \right],$$

and determine the constant $A$.

[9 marks]

Use the identity

$$e^{ikY} - e^{ikX} = ik \int_X^Y e^{ik\eta} d\eta\ .$$

to show that above result can be written in the form

$$G(x, t; 0, 0) = B \int_{x-ct}^{x+ct} d\eta\, \delta(\eta)\ ,$$

where $\delta(\eta)$ is a Dirac delta function, and determine the constant $B$. Hence sketch on an $(x, t)$ diagram the region where $G(x, t; 0, 0)$ is non-zero and give its value in this region.

[6 marks]

**3.** The Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

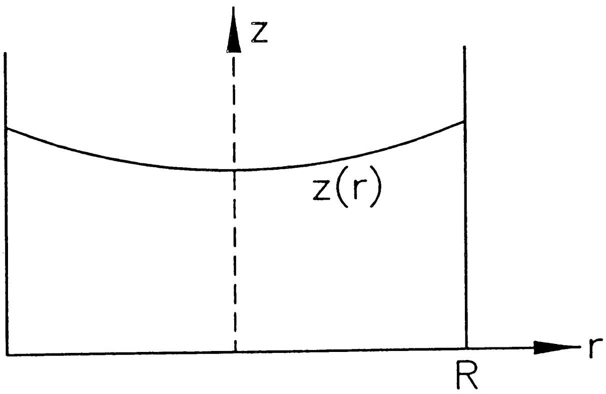
is satisfied at stationary points of a functional of the form

$$I[y] = \int_a^b dx\, F\left(y(x),\, y'(x),\, x\right) \ .$$

State, without proof, how you would find the stationary points of $I[y]$ subject to the constraint $J[y] = C$, where $C$ is a constant.

[4 marks]

A cylindrical bucket rotates about its axis with angular velocity $\omega$. A given fixed volume $V$ of water inside rotates at the same rate, adopting an equilibrium surface described by $z(r)$, as indicated in the diagram, where $z(r)$ is the height of the surface at a horizontal distance $r$ from the axis.




In a frame rotating with angular velocity $\omega$, the centrifugal potential energy of a particle at a distance $r$ from the rotation axis is given by

$$\Phi_c = -\frac{1}{2} m r^2 \omega^2 \ .$$

Hence show that, in such a frame, the combined centrifugal and gravitational potential energy of the water in the bucket is given by:

$$\Phi\,[z] = 2\pi\rho \int_0^R dr\, \left(\frac{1}{2} g r z^2(r) - \frac{1}{2}\omega^2 r^3 z(r)\right)$$

where $g$ is the standard acceleration due to gravity.

[6 marks]

By minimizing the potential energy subject to the constant volume constraint, show that the surface $z(r)$ is parabolic, i.e.

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$$z(r) = ar^2 + \frac{\lambda}{\rho g} \ .$$

Give an explicit value for the constant $a$ and determine the constant $\lambda$ in terms of the physical parameters of the problem.

[15 marks]