$$G_{mn} = g_{mn} + h_{mn}, \tag{2}$$

$$G^{mn} = g^{mn} - h^{mn} + h^{mp}h_p^n - h^{mp}h_{pq}h^{qn} + O(h^4), \tag{3}$$

$$\begin{aligned}\sqrt{G} = \sqrt{g}\Bigg[1 + \frac{1}{2}h_m^m + \frac{1}{4}\left(\frac{1}{2}\left(h_m^m\right)^2 - h^{mn}h_{mn}\right) \\ + \frac{1}{6}h_{mn}h^{np}h_p^n - \frac{1}{8}h_m^m h^{np}h_{np} + \frac{1}{48}\left(h_m^m\right)^3 + O(h^4)\Bigg].\end{aligned} \tag{4}$$

$$\Gamma^p{}_{mn} = \frac{1}{2}G^{pq}\left(G_{mq,n} + G_{nq,m} - G_{mn,q}\right), \tag{5}$$

$$\delta\Gamma^p{}_{mn} = \frac{1}{2}\left(\nabla_n h_m^p + \nabla_m h_n^p - \nabla^p h_{mn}\right), \tag{6}$$

$$\delta^2\Gamma^p{}_{mn} = -\frac{1}{2}h^{pq}\left(\nabla_n h_{mq} + \nabla_m h_{nq} - \nabla_q h_{mn}\right). \tag{7}$$

$$R^m{}_{npq} = \Gamma^m{}_{np,q} - \Gamma^p{}_{mn,p} + \Gamma^r{}_{mp}\Gamma^p{}_{nr} - \Gamma^r{}_{mn}\Gamma^p{}_{rp}, \tag{8}$$

$$\begin{aligned}\sqrt{G}\,R =& \sqrt{G}\,G^{mn}R^p{}_{mpn} \\ =& \left[\sqrt{G}\left(G^{mp}\Gamma^n{}_{mn} - G^{mn}\Gamma^p{}_{mn}\right)\right]_{,p} \\ & \sqrt{G}\,G^{mn}\left(\Gamma^p{}_{mr}\Gamma^r{}_{np} - \Gamma^p{}_{mn}\Gamma^r{}_{pr}\right),\end{aligned} \tag{9}$$

$$\mathcal{L}_{\rm grav} = \sqrt{G}\,G^{mn}\left(\Gamma^p{}_{mr}\Gamma^r{}_{np} - \Gamma^p{}_{mn}\Gamma^r{}_{pr}\right), \tag{10}$$

$$\delta^2\mathcal{L}_{\rm grav} = \sqrt{g}\left[-\frac{1}{4}(\nabla_m h_{np})^2 + \frac{1}{2}\nabla^m h^{np}\nabla_p h_{mn} - \frac{1}{2}\nabla^m h_{mn}\nabla_n h_p^p + \frac{1}{4}(\nabla_m h_n^n)^2\right], \tag{11}$$

$$\begin{aligned}\delta^3\mathcal{L}_{\rm grav} =& -\frac{1}{4}\sqrt{g}\,h^{mn}\Big[\nabla_m h_{pr}\left(4\nabla^p h_n^r - \nabla_n h^{pr}\right) + \nabla_m h_p^p\left(-2\nabla^r h_{nr} + \nabla_n h_r^r\right) \\ & - 2\nabla_m h_{np}\nabla^p h_r^r + 2\nabla^p h_{mn}\left(\nabla_p h_r^r - \nabla^r h_{pr}\right) \\ & + 2\nabla^p h_{mr}\left(\nabla_r h_{np} - \nabla_p h_n^r\right)\Big] \\ & + \frac{1}{2}h_m^m\,\delta^2\mathcal{L}_{\rm grav}.\end{aligned} \tag{12}$$