

Example 4.1

A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

Solution: In the presence of an external field \mathbf{E} , the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Fig. 4.2. (Because the actual displacements

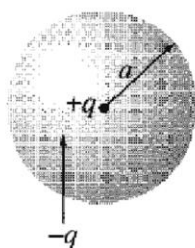


Figure 4.1

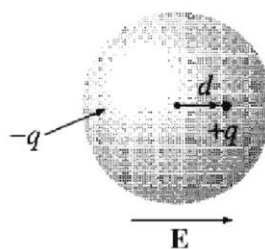


Figure 4.2

involved are extremely small, as you'll see in Prob. 4.1, it is reasonable to assume that the electron cloud retains its spherical shape.) Say that equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$, where E_e is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

(Prob. 2.12). At equilibrium, then,

$$E = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}, \quad \text{or } p = qd = (4\pi\epsilon_0 a^3)E.$$

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v, \quad (4.2)$$

where v is the volume of the atom. Although this atomic model is extremely crude, the result (4.2) is not too bad—it's accurate to within a factor of four or so for many simple atoms.

Problem 4.2

First find the field, at radius r , using Gauss' law: $\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$, or $E = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} Q_{\text{enc}}$.

$$\begin{aligned} Q_{\text{enc}} &= \int_0^r \rho \, d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2\bar{r}/a} \bar{r}^2 \, d\bar{r} = \frac{4q}{a^3} \left[-\frac{a}{2} e^{-2\bar{r}/a} \left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2} \right) \right]_0^r \\ &= -\frac{2q}{a^2} \left[e^{-2r/a} \left(r^2 + ar + \frac{a^2}{2} \right) - \frac{a^2}{2} \right] = q \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right]. \end{aligned}$$

[Note: $Q_{\text{enc}}(r \rightarrow \infty) = q$.] So the field of the electron cloud is $E_e = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right]$. The proton will be shifted from $r = 0$ to the point d where $E_e = E$ (the external field):

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left[1 - e^{-2d/a} \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) \right].$$

Expanding in powers of (d/a) :

$$\begin{aligned} e^{-2d/a} &= 1 - \left(\frac{2d}{a} \right) + \frac{1}{2} \left(\frac{2d}{a} \right)^2 - \frac{1}{3!} \left(\frac{2d}{a} \right)^3 + \dots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a} \right)^2 - \frac{4}{3} \left(\frac{d}{a} \right)^3 + \dots \\ 1 - e^{-2d/a} \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) &= 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a} \right)^2 - \frac{4}{3} \left(\frac{d}{a} \right)^3 + \dots \right) \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) \\ &= 1 - 1 + 2\frac{d}{a} - 2\frac{d^2}{a^2} + 2\frac{d^2}{a^2} + 4\frac{d^3}{a^3} + 4\frac{d^3}{a^3} - 2\frac{d^3}{a^3} - 4\frac{d^3}{a^3} + \frac{4}{3}\frac{d^3}{a^3} + \dots \\ &= \frac{4}{3} \left(\frac{d}{a} \right)^3 + \text{higher order terms.} \end{aligned}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right) = \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd) = \frac{1}{3\pi\epsilon_0 a^3} p. \quad \boxed{\alpha = 3\pi\epsilon_0 a^3.}$$

[Not so different from the *uniform* sphere model of Ex. 4.1 (see Eq. 4.2). Note that this result predicts $\frac{1}{4\pi\epsilon_0} \alpha = \frac{3}{4} a^3 = \frac{3}{4} (0.5 \times 10^{-10})^3 = 0.09 \times 10^{-30} \text{ m}^3$, compared with an experimental value (Table 4.1) of $0.66 \times 10^{-30} \text{ m}^3$. Ironically the "classical" formula (Eq. 4.2) is slightly *closer* to the empirical value.]