Epilogue to epsilon camp geometry notes

Volume calculations with and without calculus

Terminology: Today we speak of the interior of a sphere as a ball, so we would not speak of finding the volume of the sphere in three space, but only its surface area. I.e. we also consider the sphere in three dimensional space to be only 2 dimensional even though it is curved and lives in three space. Thus we say the area of the 2 - sphere of radius R is $4\pi R^2$ and the volume of the 3-ball is $(4/3)\pi^R^3$.

"Surface area" versus volume: If we try to compute the volume of the 3- sphere in 4 space, or the 4 dimensional volume of its interior, the 4- ball, we have a relationship between the sphere and the ball analogous to the one found by Archimedes. Just as the volume of the 3-ball equals R/3 times the surface area of the 2-sphere, the (4 dimensional) volume of the 4- ball of radius R equals R/4 times the (3 dimensional) volume of its surface, the 3- sphere. So again we only need to find one of them.

Volumes by slicing

If we try to use calculus to do this 4 dimensional volume in the same way as for the 3-ball, using volumes by slicing, an algebraic difficulty arises. The radius of the slice is always a square root, and in odd dimensions that square root is raised to an odd power, which makes it a fractional power that is harder to anti - differentiate. I.e. the slice area of the three ball is the area of a 2-ball or disc, which was $\pi r^2 = \pi(R^2-x^2)$ by Pythagoras. This is a nice integral power of x and is easy to anti - differentiate as we saw earlier. However the slice volume of a 4 ball is the volume of a 3-ball, namely $(4/3)\pi r^3 = (4/3)\pi(R^2-x^2)^{(3/2)}$, since again $r^2 = R^2-x^2$. This formula is harder to anti differentiate. Although it is possible to do it using trig functions, and you will learn this in calculus, we will use an easier approach.

Area of a 2-disc revisited:

horizontal slices: If we look back at the area of a 2-ball or disc, it is more natural to look at it as an expanding family of circles, rather than a growing stack of straight slices. I.e. if we grow the area upwards, with the slice at height x being a segment of length 2r, where $r^2 = R^2-x^2$, then we have a slice length formula $2r = 2(R^2-x^2)^{(1/2)}$, which is hard to anti - differentiate. In fact you will recall we did this area problem by taking a limit of areas of triangles rather than by calculus.

circular slices: It is easy also by calculus if we grow the area outwards, with the leading edge of the growing area being a circle of radius x. Then the slice length is the length of this circle, which is $2\pi r = 2\pi x$, and this is easy to anti differentiate, as πx^2. Setting x=R gives us πR^2 immediately as the area of the 2-disc, i.e. the area of the interior of the circle. [I am reminded of a remark long ago by a friend of mine, an Indian artist who saw me shading a disc with horizontal lines in a drawing, and said he would never do it that way, but would draw expanding circles instead. It just did not make visual sense to him my way.]

center of mass : The area calculation for a disc can also be done without calculus in a way used by the Greek mathematician Pappus, and understood by Archimedes, as follows. One knows in physics that the momentum of a body can be computed by thinking of the entire body as located at one point, its center of mass, or center of gravity. A 2-disc is generated by revolving one radius around the center of the circle. The area generated is equal to the length of the radius multiplied by the distance traveled by its center of mass, or center of area. Since the center of the radius of length R is the point at distance R/2 from the center, when the radius revolves around the center that point travels a distance of $2\pi(R/2) = \pi R$. Since the radius has length R, thus the area generated equals πR^2.

This calculation is essentially the same as the one we did earlier using limits of triangles. Recall that calculation yielded the formula $A = (1/2)CR$, where C is the circumference of the circle or the limit of the bases of the triangles. Another way to look at the area formula for a triangle is that it equals the height times the average base which is B/2, the length of a segment parallel to the base but halfway up the triangle. Then C/2 is the limiting value of the total average base of the triangles approximating the circle, so the formula for the area of the circle again equals $(C/2)R$. This of course equals the distance traveled by the center of the revolving radius times its length.

Volume of a 3-ball revisited:

i) horizontal slices: For the volume of the 3-ball the opposite algebraic situation occurs. I.e. if we think of the 3-ball in terms of horizontal slices, the formula $x^2 + y^2 + z^2 = R^2$ for the 3- ball, gives at height x, the formula $y^2+w^2 = R^2-x^2 = r^2$, the 2-disc of radius $r = \sqrt{R^2-x^2}$ as we know. Then the slice area is $\pi r^2 = \pi(R^2-x^2)$, which is easy to anti-differentiate, as we did before, getting $\pi(R^2x - x^3/3)$. Setting x=R of course gives (half) the volume of the 3-ball as $(2\pi/3)R^3$.

ii) cylindrical slices ("shells"): We can also look at a 3-ball as obtained by revolving the right half of a 2-disc around the y axis. If we think of the 2-disc as a union of vertical lines drawn from x = 0 to x = R, we have a line of height H, where $H^2 = R^2-x^2$. Thus the revolved segment sweeps out a cylindrical slice of the 3 ball having area $2\pi rH = 2\pi x.\sqrt{R^2-x^2}$. This formula is harder to anti - differentiate than the horizontal slice formula above, but not too hard using the "chain rule" or "substitution" formula which one learns in calculus.

iii) center of mass: We can calculate the volume of a ball in principle by multiplying the area of the half disc, by the distance traveled by its center of mass. However it is not at all obvious where the center of mass is for a half disc. Since we already know the volume of 3-ball, we can use it backwards to locate that center of mass as follows. If the center of mass of the right half of the disc of radius R is located at distance r from the y axis, then the volume $(4/3)\pi R^3$ of the 3-ball, equals the distance $2\pi r$ traveled by this point times the area $(1/2)\pi R^2$ of the half disc. Thus we should have $2\pi r(1/2)\pi R^2 = (4/3)\pi R^3$. This implies, let's see now, $\pi^2.r.R^2 = (4/3)\pi R^3$, so $r = 4R/(3\pi)$ I hope, or a little closer than halfway to the y axis. This makes sense because the half disc is thicker near the y axis.

Since Archimedes knew the volume of 3-ball, he would have known this center of mass as well.

Volume of a 4 dimensional ball

horizontal slices: If we consider the 4-ball of radius R, with equation $x^2+y^2+z^2+w^2 = R^2$, the horizontal slice at height x is the 3-ball with equation $y^2 + z^2 + w^2 = (R^2-x^2)$, of radius $r = \mathrm{sqrt}(R^2-x^2)$. (As usual we only consider half the 4-ball, starting with the slice at height $x = 0$ being the 3-dimensional "hemisphere" $y^2 + z^2 + w^2 = R^2$.) Here again, since 3 is an odd dimension our slice volume formula equals $(4/3)\pi r^3 = (4/3)\pi(R^2-x^2)^{(3/2)}$, which is again hard to anti-differentiate without using complicated trig formulas and a new technique of "integration by parts".

"Cylindrical shells": Just as in all previous cases, we can generate a 4-ball by revolving half a 3-ball around an axis. Although is hard to visualize, we proceed exactly by analogy, and consider revolving the horizontal slices of that half 3-ball around an axis. Thus the slice at height x would be a 2 disc of radius r, where $r^2 = R^2-x^2$ as before, and this 2 dimensional slice would revolve around a circular path of length $2\pi x$. Thus the full revolved 3 dimensional cylindrical slice of the 4 – ball would have volume $2\pi x(\pi r^2) = 2\pi^2.x.(R^2-x^2) = 2\pi^2 R^2.x - 2\pi^2.x^3$. This formula is easy to anti-differentiate using the familiar formula $x^{(n+1)}/(n+1)$ for the anti-derivative of x^n, yielding $\pi^2 R^2.x^2 - (1/2)\pi^2 x^4$. Setting $x = R$, gives the volume of the 4-ball as $(1/2)\pi^2.R^4$.

Any calculus student could do this calculation but most have not seen it.

Centers of mass: Again we would not know the center of mass of half a 3-ball without using the volume formula above for a 4-ball, so cannot yet compute that volume this way, but we can use Archimedes' trick to do a calculation that Archimedes could have done. Namely he showed that the volume of half a 3-ball equals the difference of the volumes of a cylinder min us that of a cone. Now the center of mass of a cylinder is obviously half way up, and Archimedes knew that just as the center of mass of a triangle is 1/3 of the way up from the base, the center of mass of a cone is ¼ the way up from the base.

Thus we can use centers of mass and subtraction to get the volume of a 4-ball. I.e. a cylinder of height R and base radius R has center of mass at height R/2, and volume

$\pi R^2.R$, so revolving it around an axis at its base gives 4 dimensional volume of $2\pi(R/2).\pi R^2.R = \pi^2.R^4$. Now the inverted cone of height R and base radius R has center of mass at distance ¼ of the way from its base, hence distance $(3R/4)$ from the axis, and volume $(1/3)\pi R^2.R$. Thus revolving it generates a 4 dimensional volume equal to $(2\pi)(3R/4). (1/3)\pi R^2.R = (1/2)\pi^2.R^4$. Subtracting the volume of the revolved cone from that of the revolved cylinder, gives the 4 dimensional volume of the revolved half 3-ball, i.e. the volume of the full 4-ball as $\pi^2.R^4 - (1/2)\pi^2.R^4 = (1/2)\pi^2.R^4$.

Remarks: This is another little calculation I made up just for you guys. I.e. I have not seen how to generalize Archimedes' work to 4 dimensions before, although it is probably out there somewhere in the wide wide world.

The only thing I have not justified is how Archimedes knew the center of mass of a cone. He probably discovered it using a balance beam, and then justified it later mathematically. One approach is to take the limit of an approximation by slabs as follows. Chopping an inverted cone of height R and base radius R, into n horizontal slabs and approximating those slabs by discs, gives n discs of base radii $(1/n)R$, $(2/n)R, \ldots, (n/n)R$, and all of height R/n. The disc of radius kR/n generates force on the balance beam proportional to its distance kR/n from the fulcrum or balance point, which causes it to try to revolve around a circle also of radius kR/n.

Thus the disc of radius kR/n, and volume $(R/n)(\pi k^2R^2/n^2)$ would generate force or work or something proportional to $(kR/n)\ (R/n)(\pi k^2R^2/n^2)$. Not being a physics guy, I prefer to multiply this by 2π and think of it as generating 4 dimensional volume of $(2\pi kR/n)\ (R/n)(\pi k^2R^2/n^2) = 2\pi^2.k^3.R^4/n^3$.

Adding up over all n of these slabs for $k=1,\ldots,n$, gives total 4 dimensional volume of $(1/n^4)[2\pi^2.R^4]/(1^3+2^3+\ldots+n^3)$. Now we know, and maybe Archimedes did too, a formula for the sum of those cubes of form $n^4/4$ + lower degree terms. Thus the formula becomes $(1/4)(2\pi^2R^4)(1/n^4)(n^4 +$ terms of degree 3 or less in n). As $n\rightarrow$infinity, this approaches the limit $(1/2)(\pi^2R^4)$. To get this same result by revolving a mass of the same volume as the cone and concentrated at one point at distance r from the axis we would need $(2\pi r)(1/3)(\pi R^3) = (1/2)(\pi^2R^4)$. Solving for r gives $r = (3/4)R$ (measured from the vertex of the cone), as we claimed.

Volume of the 3-sphere, i.e. "surface area of the 4-ball"

To get the 3-dimensional volume of the surface of the 4-ball we can use Archimedes' relation and just multiply the 4-dimensional volume of the ball by $4/R$. This gives $2\pi^2R^3$ as the surface "area" of the 4-ball, i.e. for the 3-dimensional volume of the 3-sphere.