7. Evaluate: $I = \int_0^\infty \mathrm{d}x \, \frac{x^\alpha}{x^2 + b^2} \qquad 0 < \alpha < 1$

Solution (Incorrect)
We begin with

$$\oint_C \frac{z^\alpha}{z^2 + b^2} \, \mathrm{d}z, \qquad 0 < \alpha < 1$$

with C shown in Figure 2.

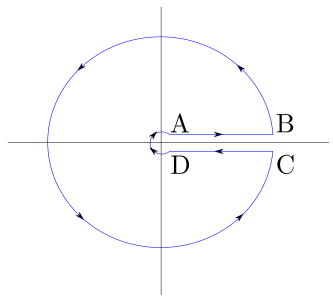

Figure 2: The contour we'll integrate.

Caution, the integrand *must* be single valued, and we guarantee this by limiting our domain. The branch cut will be along the positive real axis. The origin is our branch point. The integrand is meromorphic now (analytic except where it has poles); so we can proceed as normal. We have roots at $z = \pm ib$.

At $z = \mathrm{i}b$, $$\operatorname*{Res}_{z=\mathrm{i}b} \left(\frac{z^\alpha}{z^2 + b^2} \right) = \frac{(\mathrm{i}b)^\alpha}{\mathrm{i}b + \mathrm{i}b} = \frac{1}{2} \, (\mathrm{i}b)^{\alpha - 1}$$

At $z = -\mathrm{i}b$, $$\operatorname*{Res}_{z=-\mathrm{i}b} \left(\frac{z^\alpha}{z^2 + b^2} \right) = \frac{(-\mathrm{i}b)^\alpha}{-\mathrm{i}b - \mathrm{i}b} = \frac{1}{2} \, (-\mathrm{i}b)^{\alpha - 1}$$

Then,

$$\oint_C \frac{z^\alpha}{z^2 + b^2} \, \mathrm{d}z = 2\pi\mathrm{i} \left(\frac{1}{2} \, (\mathrm{i}b)^{\alpha - 1} - \frac{1}{2} \, (-\mathrm{i}b)^{\alpha - 1} \right) = \mathrm{i}\pi \left[(\mathrm{i}b)^{\alpha - 1} + (-\mathrm{i}b)^{\alpha - 1} \right]$$

Along the inner and outer circle, the integral tends to zero since it takes the following form:

$$\int \frac{\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^\alpha}{\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^2 + b^2} \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta = \mathrm{i} \int \frac{\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^{\alpha+1}}{\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^2 + b^2} \, \mathrm{d}\theta$$

The denominator dominates when $r \to \infty$, and the numerator goes to zero when $r \to 0$. Now we must consider the integrals along AB and CD. Along AB, $\theta = 0$; so $z = x\mathrm{e}^{\mathrm{i}\cdot 0} = x$. Along CD, $\theta = 2\pi$; so $z = x\mathrm{e}^{2\pi\mathrm{i}}$ and $\mathrm{d}z = \mathrm{e}^{2\pi\mathrm{i}} \, \mathrm{d}x = \mathrm{d}x$.

$$\underbrace{\int_0^\infty}_{AB} \frac{x^\alpha}{x^2 + b^2} \, \mathrm{d}x \qquad \text{and} \qquad \underbrace{\int_\infty^0}_{CD} \frac{\left(x\mathrm{e}^{2\pi\mathrm{i}}\right)^\alpha}{x^2\mathrm{e}^{2\pi\mathrm{i}} + b^2} \, \mathrm{e}^{2\pi\mathrm{i}} \, \mathrm{d}x = -\mathrm{e}^{2\pi\mathrm{i}\alpha} \int_0^\infty \frac{x^\alpha}{x^2 + b^2} \, \mathrm{d}x$$

Adding all the non-zero integrations together and combining our previous result,

$$\int_0^\infty \frac{x^\alpha}{x^2 + b^2} \, \mathrm{d}x = \frac{\mathrm{i}\pi \left[(\mathrm{i}b)^{\alpha - 1} + (-\mathrm{i}b)^{\alpha - 1} \right]}{1 - \mathrm{e}^{2\pi\mathrm{i}\alpha}}$$

■