(*a*) *If* $x \in (0,1)$, *then* $x \in E + r$ *for some rational* $r \in (-1,1)$.
(*b*) *If* r *and* s *are distinct rationals, then* $(E + r) \cap (E + s) = \emptyset$.

To prove (*a*), note that to every $x \in (0,1)$ there corresponds a $y \in E$ such that $x \sim y$. If $r = x - y$, then $x = y + r \in E + r$.

To prove (*b*), suppose $x \in (E + r) \cap (E + s)$. Then $x = y + r = z + s$ for some $y \in E$, $z \in E$. Since $y - z = s - r \neq 0$, we have $y \sim z$, and E contains two equivalent points, in contradiction to our choice of E.

Now assume that E is Lebesgue measurable, and put $\alpha = m(E)$. Define $S = \bigcup(E + r)$, the union being extended over all rational $r \in (-1,1)$. By (*b*), the sets $E + r$ are pairwise disjoint; since m is translation invariant, $m(E + r) = \alpha$ for every r; since $S \subset (-1,2)$, $m(S) \leq 3$. The countable additivity of m now forces $\alpha = 0$, and hence $m(S) = 0$. But (*a*) implies that $(0,1) \subset S$, hence $1 \leq m(S)$, and we have a contradiction.

Continuity Properties of Measurable Functions

Since the continuous functions played such a prominent role in our construction of Borel measures, and of Lebesgue measure in particular, it seems reasonable to expect that there are some interesting relations between continuous functions and measurable functions. In this section we shall give two theorems of this kind.

We shall assume, in both of them, *that* μ *is a measure on a locally compact Hausdorff space* X *which has the properties stated in Theorem* 2.14. In particular, μ could be Lebesgue measure on some R^k.

2.23 Lusin's Theorem *Suppose* f *is a complex measurable function on* X, $\mu(A) < \infty$, $f(x) = 0$ *if* $x \notin A$, *and* $\epsilon > 0$. *Then there exists a* $g \in C_c(X)$ *such that*

$$\mu(\{x: f(x) \neq g(x)\}) < \epsilon. \tag{1}$$

Furthermore, we may arrange it so that

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|. \tag{2}$$

PROOF Assume first that $0 \leq f < 1$ and that A is compact. Attach a sequence $\{s_n\}$ to f, as in the proof of Theorem 1.17, and put $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $2^n t_n$ is the characteristic function of a set $T_n \subset A$, and

$$f(x) = \sum_{n=1}^{\infty} t_n(x) \qquad (x \in X). \tag{3}$$

Fix an open set V such that $A \subset V$ and $\bar{V}$ is compact. There are compact sets K_n and open sets V_n such that $K_n \subset T_n \subset V_n \subset V$ and

Put $c = \mu(Q_0)$, where Q_0 is a 1-box. Since Q_0 is the union of 2^{nk} disjoint 2^{-n}-boxes, since these are translates of each other, and since $m(Q_0) = 1$, we have

$$(7) \qquad 2^{nk}\mu(Q) = \mu(Q_0) = cm(Q_0) = 2^{nk}cm(Q)$$

for every 2^{-n}-box Q. Property 2.19(d) now implies that $\mu(E) = cm(E)$ for every open set E, and the regularity of m and μ (Theorem 2.18) shows that this last equation holds for every Borel set E.

This completes the proof.

2.21 Remarks If m is the Lebesgue measure on R^k, it is customary to write $L^1(R^k)$ in place of $L^1(m)$. If E is a Lebesgue measurable subset of R^k, and if m is restricted to the measurable subsets of E, a new measure space is obtained in an obvious fashion. The phrase "$f \in L^1$ on E" or "$f \in L^1(E)$" is used to indicate that f is integrable on this measure space.

If $k = 1$, if I is any of the sets (a,b), $(a,b]$, $[a,b)$, $[a,b]$, and if $f \in L^1(I)$, it is customary to write

$$\int_a^b f(x)\, dx \qquad \text{in place of} \qquad \int_I f\, dm.$$

Since the Lebesgue measure of any single point is 0, it makes no difference over which of these four sets the integral is extended.

If f is a continuous complex function on $[a,b]$, then the Riemann integral of f and the Lebesgue integral of f over $[a,b]$ coincide. This is obvious from our construction if $f(a) = f(b) = 0$ and if $f(x)$ is defined to be 0 for $x < a$ and for $x > b$. The general case follows without difficulty. Actually the same thing is true for every Riemann integrable f on $[a,b]$. Since we shall have no occasion to discuss Riemann integrable functions in the sequel, we omit the proof and refer to Theorem 10.33 of [26].

A natural question, which may have occurred to some readers, is whether *every* subset of R^k is Lebesgue measurable. It is a consequence of the axiom of choice that the answer is negative, even for $k = 1$.

2.22 Example For real numbers x and y, write $x \sim y$ if and only if $x - y$ is rational. It is clear that $x \sim x$, that $x \sim y$ implies $y \sim x$, and that $x \sim y$, $y \sim z$ implies $x \sim z$. Thus $\sim$ is an equivalence relation. (In algebraic terminology, letting Q be the additive group of the rational numbers, each equivalence class is a coset of Q in R^1.) Let E be a set in (0,1) which contains exactly one point in every equivalence class. (The assertion that there is such a set E is a direct application of the axiom of choice.) *We claim that E is not Lebesgue measurable.*

As in Sec. 2.19, let $E + r = \{x + r : x \in E\}$. We need two properties of E:

Now suppose $f \in C_c(R^k)$, f is real, W is an open k-cell which contains the support of f, and $\epsilon > 0$. The uniform continuity of f ([26], Theorem 4.19) shows that there is an integer N and that there are functions g and h with support in W, such that (i) g and h are constant on each box belonging to Ω_N, (ii) $g \leq f \leq h$, and (iii) $h - g < \epsilon$. If $n > N$, Property 2.19(c) shows that

$$(2) \qquad \Lambda_N g = \Lambda_n g \leq \Lambda_n f \leq \Lambda_n h = \Lambda_N h.$$

Thus the upper and lower limits of $\{\Lambda_n f\}$ differ by at most ϵ vol (W), and since ϵ was arbitrary, we have proved the existence of

$$(3) \qquad \Lambda f = \lim_{n \to \infty} \Lambda_n f \qquad (f \in C_c(R^k)).$$

It is immediate that Λ is a positive linear functional on $C_c(R^k)$. (In fact, Λf is precisely the Riemann integral of f over R^k. We went through the preceding construction in order not to have to rely on any theorems about Riemann integrals in several variables.) *We define m and $\mathfrak{M}$ to be the measure and σ-algebra associated with this Λ as in Theorem* 2.14.

Since Theorem 2.14 gives us a complete measure and since R^k is σ-compact, Theorem 2.17 implies assertion (b) of Theorem 2.20.

To prove (a), let W be the open cell 2.19(4), let E_r be the union of those boxes belonging to Ω_r whose closure lies in W, and choose f so that $\bar{E}_r \prec f \prec W$. Our construction of Λf then shows that

$$(4) \qquad \Lambda f \geq \prod_{i=1}^{k} (\beta_i - \alpha_i - 2^{1-r}).$$

Let $r \to \infty$, and recall that

$$(5) \qquad m(W) = \sup \{\Lambda f : f \prec W\},$$

by the construction in Theorem 2.14. Thus $m(W) = \text{vol}\,(W)$ for every open cell W, and since every cell is the intersection of a decreasing sequence of open cells, we obtain (a).

Since vol $(W + x) =$ vol (W), it follows that

$$(6) \qquad m(E + x) = m(E) \qquad (x \in R^k)$$

holds for every cell E; in particular, (6) holds for every box E; Property 2.19(d) therefore implies that (6) holds for every open set E; and now (6) follows for every $E \in \mathfrak{M}$, since

$$m(E) = \inf \{m(V) : E \subset V,\ V \text{ open}\}.$$

This proves (c).

Finally, suppose μ is a translation invariant Borel measure on R^k.

2^{-n}-boxes with corners at points of P_n. We shall need the following four properties of $\{\Omega_n\}$. The first three are obvious by inspection.

(a) *If n is fixed, each $x \in R^k$ lies in one and only one member of Ω_n.*
(b) *If $Q' \in \Omega_n$, $Q'' \in \Omega_r$, and $r < n$, then either $Q' \subset Q''$ or $Q' \cap Q'' = \varnothing$.*
(c) *If $Q \in \Omega_r$, then* vol $(Q) = 2^{-rk}$; *and if $n > r$, the set P_n has exactly $2^{(n-r)k}$ points in Q.*
(d) *Every nonempty open set in R^k is a countable union of disjoint boxes belonging to $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \cdots$.*

PROOF OF (*d*) If V is open, every $x \in V$ lies in an open ball which lies in V; hence $x \in Q \subset V$ for some Q belonging to some Ω_n. In other words, V is the union of all boxes which lie in V and which belong to some Ω_n. From this collection of boxes, select those which belong to Ω_1, and remove those in $\Omega_2, \Omega_3, \ldots$ which lie in any of the selected boxes. From the remaining collection, select those boxes of Ω_2 which lie in V, and remove those in $\Omega_3, \Omega_4, \ldots$ which lie in any of the selected boxes. If we proceed in this way, (*a*) and (*b*) show that (*d*) holds.

2.20 Theorem *There exists a positive complete measure m defined on a σ-algebra $\mathfrak{M}$ in R^k, with the following properties:*

(a) $m(W) =$ vol (W) *for every k-cell W.*
(b) *$\mathfrak{M}$ contains all Borel sets in R^k; more precisely, $E \in \mathfrak{M}$ if and only if there are sets A and $B \subset R^k$ such that $A \subset E \subset B$, A is an F_σ, B is a G_δ, and $m(B - A) = 0$. Also, m is regular.*
(c) *m is translation invariant, i.e.,*

$$m(E + x) = m(E)$$

for every $E \in \mathfrak{M}$ and every $x \in R^k$.
(d) *If μ is any positive translation invariant Borel measure on R^k such that $\mu(K) < \infty$ for every compact set K, then there is a constant c such that $\mu(E) = cm(E)$ for all Borel sets $E \subset R^k$.*

The members of $\mathfrak{M}$ are the *Lebesgue measurable* sets in R^k; m is the *Lebesgue measure* on R^k. When clarity requires it, we shall write m_k in place of m. For a description of other measures on R^1, see Theorem 8.14.

PROOF If f is any complex function on R^k, with compact support, define

$$\Lambda_n f = 2^{-nk} \sum_{x \in P_n} f(x) \qquad (n = 1, 2, 3, \ldots), \tag{1}$$

where P_n is as in Sec. 2.19.

$F \subset E \subset V$ and $\mu(V - F) < \epsilon$. But $V - F$ is open. Hence (3) shows that $\lambda(V - F) < \epsilon$, and this proves the regularity of λ, as in Theorem 2.17.

Note: It also follows easily that $\lambda(E) = \mu(E)$ for every Borel set E in X.

In Exercise 17 a compact Hausdorff space is described which contains an open set which is not σ-compact and in which the preceding theorem fails.

Lebesgue Measure

2.19 Euclidean Spaces Euclidean k-dimensional space R^k is the set of all points $x = (\xi_1, \ldots, \xi_k)$ whose coordinates ξ_i are real numbers, with the following algebraic and topological structure:

If $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k)$, and α is a real number, $x + y$ and αx are defined by

$$(1) \qquad x + y = (\xi_1 + \eta_1, \ldots, \xi_k + \eta_k), \qquad \alpha x = (\alpha\xi_1, \ldots, \alpha\xi_k).$$

This makes R^k into a real vector space. If $x \cdot y = \Sigma \xi_i \eta_i$ and $|x| = (x \cdot x)^{\frac{1}{2}}$, the Schwarz inequality $|x \cdot y| \leq |x|\,|y|$ leads to the triangle inequality

$$(2) \qquad |x - y| \leq |x - z| + |z - y|;$$

hence we obtain a metric by setting $\rho(x,y) = |x - y|$. We assume that these facts are familiar to the reader, and shall prove them in greater generality in Chap. 4.

If $E \subset R^k$ and $x \in R^k$, the *translate of E by x* is the set

$$(3) \qquad E + x = \{y + x : y \in E\}.$$

A set of the form

$$(4) \qquad W = \{x : \alpha_i < \xi_i < \beta_i,\ 1 \leq i \leq k\},$$

or any set obtained by replacing any or all of the $<$ signs in (4) by $\leq$, is called a *k-cell;* its volume is defined to be

$$(5) \qquad \text{vol}\,(W) = \prod_{i=1}^{k} (\beta_i - \alpha_i).$$

If $a \in R^k$ and $\delta > 0$, we shall call the set

$$(6) \qquad Q(a;\delta) = \{x : \alpha_i \leq \xi_i < \alpha_i + \delta,\ 1 \leq i \leq k\}$$

the *δ-box with corner at a.* Here $a = (\alpha_1, \ldots, \alpha_k)$.

For $n = 1, 2, 3, \ldots$, we let P_n be the set of all $x \in R^k$ whose coordinates are integral multiples of 2^{-n}, and we let Ω_n be the collection of all