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"Read Euler: he is our master in everything." --PS Laplace

Definition 1.0: $x! := \prod_{q=1}^{x} q$

It was Leonhard Euler who, at the age of 22, was the first to extend the domain of $x!$ (the factorial) from non-negative integer arguments to all complex arguments less the negative integers. The analytic continuation of $x!$ was a problem likely suggested to Euler by either his then friend, colleague, and roommate Daniel Bernoulli, or by another colleague of Euler's at the St. Petersburg Academy of Science: Christian Goldbach. Insofar as one can be certain, the later was to whom, Euler, in his letter of October 13, 1729, disclosed his solution.

One way to achieve Euler's result is to observe that $\forall x, \lambda \in \mathbb{Z}^* := \mathbb{Z}^+ \cup \{0\}$ one may write

$$x! := \prod_{q=1}^{x} q = \left(\prod_{q=1}^{x} q\right) \underbrace{\lim_{\lambda\to\infty}\left(\prod_{j=x+1}^{x+\lambda} j\right)\left(\prod_{\rho=x+1}^{x+\lambda} \rho\right)^{-1}}_{=1} = \lim_{\lambda\to\infty}\left(\prod_{j=1}^{x+\lambda} j\right)\left[\prod_{k=1}^{\lambda}(x+k)\right]^{-1} = \lim_{\lambda\to\infty}\left(\prod_{k=1}^{\lambda} k\right)\left(\prod_{j=\lambda+1}^{x+\lambda} j\right)\prod_{k=1}^{\lambda}\left(\frac{1}{x+k}\right)$$

$$= \lim_{\lambda\to\infty}\lambda!\prod_{q=1}^{x}(\lambda+q)\prod_{k=1}^{\lambda}\left(\frac{1}{x+k}\right) = \lim_{\lambda\to\infty}\lambda!\left[\lambda^x\prod_{q=1}^{x}\left(\frac{\lambda+q}{\lambda}\right)\right]\prod_{k=1}^{\lambda}\left(\frac{1}{x+k}\right) = \lim_{\lambda\to\infty}\lambda!\lambda^x\prod_{k=1}^{\lambda}\left(\frac{1}{x+k}\right) =: \Gamma(x+1),$$

yielding the so-called *Euler limit form of the gamma function*, viz.

Definition 1.1 (Euler 1729): $\forall x \notin \mathbb{Z}^- \cup \{0\},\ \Gamma(x) := \lim_{\lambda\to\infty}\frac{\lambda!\lambda^{x-1}}{x(x+1)\cdots(x+\lambda-1)}$, a meromorphic function of the complex variable x which possesses simple poles at non-positive integer arguments.

From Definition 1.1 it follows that a *difference equation satisfied by the gamma function* is

Equation 1.2: $\forall x \notin \mathbb{Z}^- \cup \{0\},\ \Gamma(x+1) = x\,\Gamma(x)$.

Another *elementary property of the gamma function* whose proof is left as an exercise for the reader is given by

Equation 1.3: $\forall n \in \mathbb{Z}^*,\ \Gamma(n+1) = n!$.

That the *Euler product form of the gamma function* is often taken to be the definition thereof notwithstanding[†], in this work the author shall have the occasion to call it

Theorem 1.1 (Euler 1729, and Gauss 1811): $\forall z \notin \mathbb{Z}^- \cup \{0\},\ \Gamma(z) = \frac{1}{z}\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^z \Big/ \left(1+\frac{z}{\rho}\right)\right]$.

Proof: $$\Gamma(z) := \lim_{\lambda\to\infty}\frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-1)} = \lim_{\lambda\to\infty}\frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-1)}\cdot\underbrace{\lim_{\lambda\to\infty}\frac{\lambda}{z+\lambda}}_{=1} = \lim_{\lambda\to\infty}\frac{\lambda!\lambda^{z}}{z(z+1)\cdots(z+\lambda)}$$

$$= \frac{1}{z}\lim_{\lambda\to\infty}\lambda^z\prod_{\rho=1}^{\lambda}\left(\frac{\rho}{z+\rho}\right) = \frac{1}{z}\lim_{\lambda\to\infty}\left[\frac{\lambda(\lambda+1)!}{(\lambda+1)\lambda!}\right]^z\prod_{\rho=1}^{\lambda}\left(\frac{\rho}{z+\rho}\right) = \frac{1}{z}\lim_{\lambda\to\infty}\left[\frac{\lambda}{\lambda+1}\prod_{k=1}^{\lambda}\left(\frac{k+1}{k}\right)\right]^z\prod_{\rho=1}^{\lambda}\left(\frac{\rho}{z+\rho}\right)$$

$$= \frac{1}{z}\lim_{\lambda\to\infty}\left(1-\frac{1}{\lambda+1}\right)^z\prod_{\rho=1}^{\lambda}\left[\left(1+\frac{1}{\rho}\right)^z\left(\frac{\rho}{z+\rho}\right)\right] = \frac{1}{z}\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^z \Big/ \left(1+\frac{z}{\rho}\right)\right].$$

Not but thirteen weeks after the first, Christian Goldbach received a another letter from Euler, this one dated January 8, 1730, containing yet another definition of the gamma function, being convergent for arguments with non-negative real parts, was originally known as an *Eulerian Integral of the Second Kind*, was later dubbed the Gamma Integral by Gauss, yet is here to be humbly re-titled:

Theorem 1.2 (Euler 1730): $\Gamma(z) = \int_{t=0}^{\infty} e^{-t}t^{z-1}dt,\ \Re[z] > 0$.

[†] Gauss did, for example. [a]

Proof: From the definition of the Gamma function, make the following observation:

$$\Gamma(z) := \lim_{\lambda\to\infty} \frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-1)} = \lim_{\lambda\to\infty} \frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-2)} \int_{x=0}^{1} (1-x)^{z+\lambda-2}\, dx.$$

Now, integrate by parts.

$$\begin{aligned}\Gamma(z) &= \lim_{\lambda\to\infty} \frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-2)} \left\{ \left[x(1-x)^{z+\lambda-2} \Big|_{x=0}^{1} + (z+\lambda-2)\int_{x=0}^{1} x(1-x)^{z+\lambda-3}\, dx \right.\right\} \\ &= \lim_{\lambda\to\infty} \frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-3)} \int_{x=0}^{1} x(1-x)^{z+\lambda-3}\, dx.\end{aligned}$$

In general, k iterations of integration by parts gives $\Gamma(z) = \lim_{\lambda\to\infty} \frac{\lambda!\lambda^{z-1}}{z(z+1)\cdots(z+\lambda-k-2)k!} \int_{x=0}^{1} (1-x)^{z+\lambda-k-2} x^k dx$; in particular, $(\lambda-1)$ iterations of integration by parts yields $\Gamma(z) = \lim_{\lambda\to\infty} \lambda^z \int_{x=0}^{1} (1-x)^{z-1} x^{\lambda-1} dx$. Substitute $x^\lambda = y \Rightarrow \lambda x^{\lambda-1}dx = dy$, to get $\Gamma(z) = \lim_{\lambda\to\infty} \lambda^{z-1} \int_{y=0}^{1} \left(1-y^{\frac{1}{\lambda}}\right)^{z-1} dy$. Set $\lambda = \frac{1}{\eta}$, so that $\eta \to 0^+$ as $\lambda\to\infty$ and $\Gamma(z) = \lim_{\eta\to 0^+} \int_{y=0}^{1} \left(\frac{1-y^\eta}{\eta}\right)^{z-1} dy \overset{H}{=} \int_{y=0}^{1} \ln^{z-1}\left(\frac{1}{y}\right) dy$, where $\overset{H}{=}$ denotes the use of l'Hospital's Rule. Substitute $y = e^{-t} \Rightarrow dy = -e^{-t}dt$, to get $\Gamma(z) = \int_{t=0}^{\infty} e^{-t}t^{z-1}dt$, and the theorem is demonstrated.

Weierstrass took as the definition of the gamma function its canonical infinite product representation, the so-called Weierstrass product form of the gamma function. The desired representation of the gamma function is here obtained as a corollary to the *Weierstrass Factor Theorem*‡, the proof of which shall not be reproduced here††.

Theorem 1.3 (Weierstass): Let $f(z)$ be an entire (i.e. everywhere analytic) function with simple zeros at $z = a_1, a_2, a_3, \ldots$ where $0 < |a_1| < |a_2| < |a_3| < \cdots$ and $\lim_{M\to\infty} a_M = \infty$, then $f(z) = f(0) e^{\frac{f'(0)}{f(0)}z} \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k}} \right]$.

The *Weierstrass product form of the gamma function* is then given by

Corollary 1.1 (Weierstass): $\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{\lambda=1}^{\infty} \left[\left(1+\frac{z}{\lambda}\right) e^{-z/\lambda} \right]$ where is γ *Euler's Constant.*

Proof: Let $f(z) = \frac{1}{\Gamma(z+1)}$ so that $f(z)$ is an entire function with simple zeros at $z = a_k := -k \forall k \in \mathbb{Z}^+$ satisfying the hypotheses necessary to invoke Theorem 1.2, which yields $\frac{1}{\Gamma(z+1)} = e^{f'(0)z} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right) e^{-\frac{z}{k}} \right]$.

Set $z = 1$ in the above formula and take logarithms of the result to determine that

$$\begin{aligned} f'(0) &= \sum_{k=1}^{\infty} \left[\frac{1}{k} - \ln\left(1+\frac{1}{k}\right) \right] = \lim_{M\to\infty} \sum_{k=1}^{M} \left[\frac{1}{k} + \ln(k) - \ln(k+1) \right] = \lim_{M\to\infty} \left[\sum_{k=1}^{M} \left(\frac{1}{k}\right) - \ln(M+1) \right] + \ln(1) \\ &= \lim_{M\to\infty} \left[\sum_{k=1}^{M} \left(\frac{1}{k}\right) - \ln(M+1) \right] + \underbrace{\lim_{M\to\infty} \ln\left(1+\frac{1}{M}\right)}_{=\ln(1)} = \lim_{M\to\infty} \left[\sum_{k=1}^{M} \left(\frac{1}{k}\right) - \ln(M) \right] =: \gamma. \end{aligned}$$

The theorem is proved upon applying Equation 1.1 and replacing $f'(0)$ with γ, which is Euler's Constant.

--TO BE ADDED TO SECTION I---

‡ Note that said product may be derived from Definition 1.1 algebraically, an exercise which is left for the sufficiently curious reader.

†† A proof of this theorem may be found in any sufficiently interesting text on complex analysis, see for example [b, pgs. 136-9].

The polygamma function is denoted by ψ_n and is defined to be the $(n+1)^{\text{th}}$ derivative of the natural logarithm of the gamma function, viz.

Definition 1.4: $\psi_n(z) := \dfrac{d^{n+1}}{dz^{n+1}} \ln\Gamma(z)$.

Exercises I

1.1) Verify that Equation 1.1 follows from Definition 1.1.

1.2) a) Use the principal of mathematical induction and Definition 1.1 to establish Equation 1.2.

b) Why do you think mathematicians chose to adopt the convention $0! = 1$?

1.3) Use the Euler product form of the gamma function to show that $\lim\limits_{N\to\infty} \Gamma^n\left(1+\frac{k}{N}\right)\Big/\Gamma\left(1+\frac{kn}{N}\right) = 1$.

1.4) Show that $\operatorname*{Res}\limits_{x=-k} \Gamma(x) = (-1)^k/k!\,\forall k \in \mathbb{Z}^+$

1.5) Obtain the formula $\psi_n(z) = \sum\limits_{q=0}^{\infty} \dfrac{(-1)^{n+1} n!}{(z+q)^{n+1}}$ by taking as the definition of the gamma function:

a) the Euler limit form, b) the Euler product form, and c) the Weierstrass product form.

1.6) Evaluate $\int_{z=\frac{1}{2}}^{1} \psi_1(z)\,dz$

1.7) Show that Euler's Constant to four decimal places is $\gamma := \lim\limits_{M\to\infty}\left[\sum\limits_{k=1}^{M}\frac{1}{k} - \ln(M)\right] = 0.5772\ldots$.

II. The Integrals of Dirichlet; Liouville's Generalization Thereof

A result due to Dirichlet is given by

Theorem 2.1: If $\alpha_p, \beta_q, \Re[\gamma_r] > 0 \forall p,q,r$ and $V^n := \left\{(z_1,z_2,\ldots,z_n) \in \mathbb{R}^n \mid z_j \geq 0 \forall j, \sum\limits_{k=1}^{n}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k} \leq 1\right\}$, then

$\iint\limits_{V^n}\cdots\int \prod\limits_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda - 1}\right) dz_n \cdots dz_2 dz_1 = \prod\limits_{q=1}^{n}\left[\frac{\alpha_q^{\gamma_q}}{\beta_q}\Gamma\left(\frac{\gamma_q}{\beta_q}\right)\right]\Big/\Gamma\left(1+\sum\limits_{k=1}^{n}\frac{\gamma_k}{\beta_k}\right)$, where $\Re[w]$ denotes the real part of w.

Proof: The proof is by induction. Let I_n denote the integral on the left-hand side of the above equality.

(i) $V^1 = \{z_1 \in \mathbb{R} \mid 0 \leq z_1 \leq \alpha_1\}$, $I_1 = \int_{z_1=0}^{\alpha_1} z_1^{\gamma_1-1} dz_1 = \dfrac{\alpha_1^{\gamma_1}}{\gamma_1}\underbrace{\left[\frac{\gamma_1}{\beta_1}\Gamma\left(\frac{\gamma_1}{\beta_1}\right)\Big/\Gamma\left(1+\frac{\gamma_1}{\beta_1}\right)\right]}_{=1} = \dfrac{\alpha_1^{\gamma_1}}{\beta_1}\Gamma\left(\frac{\gamma_1}{\beta_1}\right)\Big/\Gamma\left(1+\frac{\gamma_1}{\beta_1}\right)$.

(ii) Assume the theorem holds for some fixed n.

$$\begin{aligned} V^{n+1} &= \left\{(z_1,z_2,\ldots,z_{n+1}) \in \mathbb{R}^{n+1} \mid z_j \geq 0 \forall j, \sum_{k=1}^{n+1}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k} \leq 1\right\} \\ &= \left\{(z_1,z_2,\ldots,z_{n+1}) \in \mathbb{R}^{n+1} \mid 0 \leq z_1 \leq \alpha_1, 0 \leq z_j \leq \alpha_j\left[1-\sum_{k=1}^{j-1}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\right]^{1/\beta_j} \forall j \geq 2\right\}, \end{aligned}$$

$$I_{n+1} = \iint\limits_{V^{n+1}}\cdots\int \prod_{\lambda=1}^{n+1} z_\lambda^{\gamma_\lambda-1} dz_{n+1}\cdots dz_2 dz_1 = \int_{z_1=0}^{\alpha_1}\int_{z_2=0}^{\alpha_2\left[1-\left(\frac{z_1}{\alpha_1}\right)^{\beta_1}\right]^{1/\beta_2}} \cdots \int_{z_{n+1}=0}^{\alpha_{n+1}\left[1-\sum_{k=1}^{n}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\right]^{1/\beta_{n+1}}} \prod_{\lambda=1}^{n+1}\left(z_\lambda^{\gamma_\lambda-1}\right) dz_{n+1}\cdots dz_2 dz_1$$

Apply the $(n+1)$ transformation equations: $z_m=\alpha_m y_m^{\frac{1}{\beta_m}}\forall m \Rightarrow dz_{n+1}\cdots dz_2dz_1=\prod_{m=1}^{n+1}\left(\frac{\alpha_m}{\beta_m}y_m^{\frac{1}{\beta_m}-1}\right)dy_{n+1}\cdots dy_2dy_1$

$$I_{n+1}=\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_{n+1}=0}^{1-\sum_{k=1}^{n}y_k}\prod_{\lambda=1}^{n+1}\left[\left(\alpha_\lambda y_\lambda^{\frac{1}{\beta_\lambda}}\right)^{\gamma_\lambda-1}\frac{\alpha_\lambda}{\beta_\lambda}y_\lambda^{\frac{1}{\beta_\lambda}-1}\right]dy_{n+1}\cdots dy_2dy_1$$

$$=\frac{\alpha_{n+1}^{\gamma_{n+1}}}{\beta_{n+1}}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_{n+1}=0}^{1-\sum_{k=1}^{n}y_k}y_{n+1}^{\frac{\gamma_{n+1}}{\beta_{n+1}}-1}\prod_{\lambda=1}^{n}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_{n+1}\cdots dy_2dy_1$$

$$=\left(\frac{\beta_{n+1}}{\gamma_{n+1}}\right)\frac{\alpha_{n+1}^{\gamma_{n+1}}}{\beta_{n+1}}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-\sum_{k=1}^{n-1}y_k}y_{n+1}^{\frac{\gamma_{n+1}}{\beta_{n+1}}}\Bigg|_{y_{n+1}=0}^{1-\sum_{k=1}^{n}y_k}\prod_{\lambda=1}^{n}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_n\cdots dy_2dy_1$$

$$=\frac{\alpha_{n+1}^{\gamma_{n+1}}}{\gamma_{n+1}}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-\sum_{k=1}^{n-1}y_k}\left(1-\sum_{k=1}^{n}y_k\right)^{\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_n\cdots dy_2dy_1$$

Noting that the variables $y_1,y_2,\ldots,y_{n-1}$ are held constant when integrating with respect to y_n, set

$$y_n=\left(1-\sum_{k=1}^{n-1}y_k\right)x\Rightarrow dy_n=\left(1-\sum_{k=1}^{n-1}y_k\right)dx$$

$$I_{n+1}=\frac{\alpha_n^{\gamma_n}\alpha_{n+1}^{\gamma_{n+1}}}{\beta_n\gamma_{n+1}}\int_{x=0}^{1}x^{\frac{\gamma_n}{\beta_n}-1}(1-x)^{\frac{\gamma_{n+1}}{\beta_{n+1}}}dx\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_{n-1}=0}^{1-\sum_{r=1}^{n-2}y_r}\left(1-\sum_{k=1}^{n-1}y_k\right)^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n-1}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_{n-1}\cdots dy_2dy_1$$

$$=\frac{\alpha_n^{\gamma_n}\alpha_{n+1}^{\gamma_{n+1}}}{\beta_n\gamma_{n+1}}\frac{\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(1+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\Gamma\left(1+\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_{n-1}=0}^{1-\sum_{r=1}^{n-2}y_r}\left(1-\sum_{k=1}^{n-1}y_k\right)^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n-1}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_{n-1}\cdots dy_2dy_1$$

$$=\frac{\alpha_n^{\gamma_n}\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(1+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\beta_n\gamma_{n+1}\Gamma\left(1+\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_{n-1}=0}^{1-\sum_{r=1}^{n-2}y_r}y_n^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}}\Bigg|_{y_n=0}^{1-\sum_{k=1}^{n-1}y_k}\prod_{\lambda=1}^{n-1}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_{n-1}\cdots dy_2dy_1$$

$$=\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)\frac{\alpha_n^{\gamma_n}\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(1+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\beta_n\gamma_{n+1}\Gamma\left(1+\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-\sum_{r=1}^{n-1}y_r}y_n^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}-1}\prod_{\lambda=1}^{n-1}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_n\cdots dy_2dy_1$$

$$=\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\gamma_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-\sum_{r=1}^{n-1}y_r}y_n^{\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dy_n\cdots dy_2dy_1$$

Apply the n transformation equations: $y_m=\left(\frac{z_m}{\alpha_m}\right)^{\beta_m}\forall m\Rightarrow dy_n\cdots dy_2dy_1=\prod_{m=1}^{n}\left[\frac{\beta_m}{\alpha_m}\left(\frac{z_m}{\alpha_m}\right)^{\beta_m-1}\right]dz_n\cdots dz_2dz_1$

$$I_{n+1}=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{z_1=0}^{\alpha_1}\int_{z_2=0}^{\alpha_2\left[1-\left(\frac{z_1}{\alpha_1}\right)^{\beta_1}\right]^{1/\beta_2}}\cdots\int_{z_n=0}^{\alpha_n\left[1-\sum_{k=1}^{n-1}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\right]^{1/\beta_n}}\left(\frac{z_n}{\alpha_n}\right)^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left[\frac{\alpha_\lambda^{\gamma_\lambda}}{\beta_\lambda}\left(\frac{z_\lambda}{\alpha_\lambda}\right)^{\beta_\lambda\left(\frac{\gamma_\lambda}{\beta_\lambda}-1\right)}\frac{\beta_\lambda}{\alpha_\lambda}\left(\frac{z_\lambda}{\alpha_\lambda}\right)^{\beta_\lambda-1}\right]dz_n\cdots dz_2dz_1$$

$$=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\alpha_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\iint_{V^n}\cdots\int z_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda-1}\right)dz_n\cdots dz_2dz_1,\text{ which, by hypothesis}$$

$$=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\alpha_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\prod_{q=1}^{n-1}\left[\frac{\alpha_q^{\gamma_q}}{\beta_q}\Gamma\left(\frac{\gamma_q}{\beta_q}\right)\right]\left(\frac{\alpha_n^{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}}{\beta_n}\right)\Gamma\left(\frac{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}{\beta_n}\right)\Bigg/\Gamma\left(1+\sum_{k=1}^{n-1}\left(\frac{\gamma_k}{\beta_k}\right)+\frac{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}{\beta_n}\right)$$

$$=\prod_{q=1}^{n+1}\left[\frac{\alpha_q^{\gamma_q}}{\beta_q}\Gamma\left(\frac{\gamma_q}{\beta_q}\right)\right]\Bigg/\Gamma\left(1+\sum_{k=1}^{n+1}\frac{\gamma_k}{\beta_k}\right)\text{ the required result.}$$

Corollary 2.1: If $\alpha_p,\beta_q,\Re[\gamma_r]>0\forall p,q,r$ and $V^n:=\left\{(z_1,z_2,\ldots,z_n)\in\mathbb{R}^n \mid z_j\geq 0\forall j,\sum_{k=1}^{n}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\leq t\right\}$, then

$$\iint_{V^n}\cdots\int\prod_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda-1}\right)dz_n\cdots dz_2dz_1=t^{\sum_{p=1}^{n}\frac{\gamma_p}{\beta_p}}\prod_{q=1}^{n}\left[\frac{\alpha_q^{\gamma_q}}{\beta_q}\Gamma\left(\frac{\gamma_q}{\beta_q}\right)\right]\Bigg/\Gamma\left(1+\sum_{r=1}^{n}\frac{\gamma_r}{\beta_r}\right).$$

Corollary 2.2: If $\alpha_p, \beta_q, \Re[\gamma_r] > 0 \forall p,q,r$ and $V^n := \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \mid z_j \geq 0 \forall j, t_1 \leq \sum_{k=1}^{n} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \leq t_2 \right\}$, then

$$\iint_{V^n} \cdots \int \prod_{\lambda=1}^{n} \left(z_\lambda^{\gamma_\lambda - 1} \right) dz_n \cdots dz_2 dz_1 = \left(t_2^{\sum_{p=1}^{n} \frac{\gamma_p}{\beta_p}} - t_1^{\sum_{p=1}^{n} \frac{\gamma_p}{\beta_p}} \right) \prod_{q=1}^{n} \left[\frac{\alpha_q^{\gamma_q}}{\beta_q} \Gamma\left(\frac{\gamma_q}{\beta_q} \right) \right] \Big/ \Gamma\left(1 + \sum_{r=1}^{n} \frac{\gamma_r}{\beta_r} \right).$$

Corollary 2.3: If $\alpha_p, \beta_q > 0 \forall p,q$ and $V^n := \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \mid z_j \geq 0 \forall j, \sum_{k=1}^{n} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \leq 1 \right\}$, then

$\text{content}_n\left(V^n\right) := \iint_{V^n} \cdots \int dz_n \cdots dz_2 dz_1 = \prod_{q=1}^{n} \left[\frac{\alpha_q}{\beta_q} \Gamma\left(\frac{1}{\beta_q} \right) \right] \Big/ \Gamma\left(1 + \sum_{k=1}^{n} \frac{1}{\beta_k} \right)$, where the formalism $\text{content}_n\left(V^n\right)$ is meant to convey that the integral yielding the value indicated is *ipso facto* to be interpreted as the length, area, volume, or content (a.k.a. hypervolume) of the region V^n according as $n = 1,2,3,$ or $n \geq 4$, respectively.

A generalization of Dirichlet's result is given by the stronger theorem of Louiville, namely

Theorem 2.2: If $\alpha_p, \beta_q, \Re[\gamma_r] > 0 \forall p,q,r$, $V^n := \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \mid z_j \geq 0 \forall j, \sum_{k=1}^{n} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \leq t \right\}$, and if F is continuous on $[0,t]$ then $\iint_{V^n} \cdots \int F\left[\sum_{i=1}^{n} \left(\frac{z_i}{\alpha_i} \right)^{\beta_i} \right] \prod_{\lambda=1}^{n} \left(z_\lambda^{\gamma_\lambda - 1} \right) dz_n \cdots dz_2 dz_1 = \dfrac{\prod_{k=1}^{n} \left[\frac{\alpha_k^{\gamma_k}}{\beta_k} \Gamma\left(\frac{\gamma_k}{\beta_k} \right) \right]}{\Gamma\left(\sum_{i=1}^{n} \frac{\gamma_i}{\beta_i} \right)} \int_{u=0}^{t} u^{\sum_{i=1}^{n} \left(\frac{\gamma_i}{\beta_i} \right) - 1} F(u)\, du$.

Proof: The proof is by induction. Let J_n denote the integral on the left-hand side of the above equality.

(i) $V^1 = \left\{ z_1 \in \mathbb{R} \mid 0 \leq \left(\frac{z_1}{\alpha_1} \right)^{\beta_1} \leq t \right\}$, $J_1 = \int_{z_1=0}^{\alpha_1 t^{\frac{1}{\beta_1}}} F\left[\left(\frac{z_1}{\alpha_1} \right)^{\beta_1} \right] z_1^{\gamma_1 - 1} dz_1$; let $z_1 = \alpha_1 u^{\frac{1}{\beta_1}} \Rightarrow dz_1 = \frac{\alpha_1}{\beta_1} u^{\frac{1}{\beta_1} - 1} du$ so that

$$J_1 = \int_{u=0}^{t} \left(\alpha_1 u^{\frac{1}{\beta_1}} \right)^{\gamma_1 - 1} F(u) \frac{\alpha_1}{\beta_1} u^{\frac{1}{\beta_1} - 1} du = \frac{\alpha_1^{\gamma_1}}{\beta_1} \int_{u=0}^{t} u^{\frac{\gamma_1}{\beta_1} - 1} F(u)\, du.$$

(ii) Assume the theorem holds for some fixed n.

$$V^{n+1} = \left\{ (z_1, z_2, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \mid z_j \geq 0 \forall j, \sum_{k=1}^{n+1} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \leq t \right\}$$

$$= \left\{ (z_1, z_2, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \mid 0 \leq z_1 \leq \alpha_1 t^{\frac{1}{\beta_1}}, 0 \leq z_j \leq \alpha_j \left[t - \sum_{k=1}^{j-1} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \right]^{1/\beta_j} \forall j \geq 2 \right\},$$

$$J_{n+1} = \iint_{V^{n+1}} \cdots \int F\left[\sum_{i=1}^{n+1} \left(\frac{z_i}{\alpha_i} \right)^{\beta_i} \right] \prod_{\lambda=1}^{n+1} \left(z_\lambda^{\gamma_\lambda - 1} \right) dz_{n+1} \cdots dz_2 dz_1$$

$$= \int_{z_1=0}^{\alpha_1 t^{\frac{1}{\beta_1}}} \int_{z_2=0}^{\alpha_2 \left[t - \left(\frac{z_1}{\alpha_1} \right)^{\beta_1} \right]^{1/\beta_2}} \cdots \int_{z_{n+1}=0}^{\alpha_{n+1} \left[t - \sum_{k=1}^{n} \left(\frac{z_k}{\alpha_k} \right)^{\beta_k} \right]^{1/\beta_{n+1}}} F\left[\sum_{i=1}^{n+1} \left(\frac{z_i}{\alpha_i} \right)^{\beta_i} \right] \prod_{\lambda=1}^{n+1} \left(z_\lambda^{\gamma_\lambda - 1} \right) dz_{n+1} \cdots dz_2 dz_1$$

Apply the $(n+1)$ transformation equations: $z_m = \alpha_m y_m^{\frac{1}{\beta_m}} \forall m \Rightarrow dz_{n+1} \cdots dz_2 dz_1 = \prod_{m=1}^{n+1} \left(\frac{\alpha_m}{\beta_m} y_m^{\frac{1}{\beta_m} - 1} \right) dy_{n+1} \cdots dy_2 dy_1$

$$J_{n+1} = \int_{y_1=0}^{t} \int_{y_2=0}^{t-y_1} \cdots \int_{y_{n+1}=0}^{t - \sum_{k=1}^{n} y_k} F\left(\sum_{i=1}^{n+1} y_i \right) \prod_{\lambda=1}^{n+1} \left[\left(\alpha_\lambda y_\lambda^{\frac{1}{\beta_\lambda}} \right)^{\gamma_\lambda - 1} \frac{\alpha_\lambda}{\beta_\lambda} y_\lambda^{\frac{1}{\beta_\lambda} - 1} \right] dy_{n+1} \cdots dy_2 dy_1$$

$$= \prod_{q=1}^{n+1} \left(\frac{\alpha_q^{\gamma_q}}{\beta_q} \right) \int_{y_1=0}^{1} \int_{y_2=0}^{1-y_1} \cdots \int_{y_{n+1}=0}^{1 - \sum_{k=1}^{n} y_k} F\left(\sum_{i=1}^{n+1} y_i \right) \prod_{\lambda=1}^{n+1} \left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda} - 1} \right) dy_{n+1} \cdots dy_2 dy_1$$

Substitute $y_{n+1} = y_n (1-u) u^{-1} \Rightarrow dy_{n+1} = -y_n u^{-2} du$

$$J_{n+1}=\prod_{q=1}^{n+1}\left(\frac{\alpha_q^{\gamma_q}}{\beta_q}\right)\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{y_n=0}^{1-\sum_{k=1}^{n-1}y_k}\int_{u=y_n\left(1-\sum_{i=1}^{n-1}y_i\right)^{-1}}^{1}F\left(\frac{y_n}{u}+\sum_{i=1}^{n-1}y_i\right)y_n^{\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)(1-u)^{\frac{\gamma_{n+1}}{\beta_{n+1}}-1}u^{-\frac{\gamma_{n+1}}{\beta_{n+1}}-1}du\,dy_n\cdots dy_2dy_1$$

Switch the order of integration from $du\,dy_n dy_{n-1}\cdots dy_1$ to $dy_n du\,dy_{n-1}\cdots dy_1$

$$J_{n+1}=\prod_{q=1}^{n+1}\left(\frac{\alpha_q^{\gamma_q}}{\beta_q}\right)\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{u=0}^{1}\int_{y_n=0}^{u\left(1-\sum_{i=1}^{n-1}y_i\right)}F\left(\frac{y_n}{u}+\sum_{i=1}^{n-1}y_i\right)y_n^{\frac{\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)(1-u)^{\frac{\gamma_{n+1}}{\beta_{n+1}}-1}u^{-\frac{\gamma_{n+1}}{\beta_{n+1}}-1}dy_n du\cdots dy_2dy_1$$

Substitute $y_n=uY\Rightarrow dy_n=udY$

$$\begin{aligned}J_{n+1}&=\prod_{q=1}^{n+1}\left(\frac{\alpha_q^{\gamma_q}}{\beta_q}\right)\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{u=0}^{1}\int_{Y=0}^{\left(1-\sum_{i=1}^{n-1}y_i\right)}F\left(Y+\sum_{i=1}^{n-1}y_i\right)Y^{\frac{\gamma_\lambda}{\beta_\lambda}+\frac{\gamma_{n+1}}{\beta_{n+1}}-1}\prod_{\lambda=1}^{n-1}\left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)(1-u)^{\frac{\gamma_{n+1}}{\beta_{n+1}}-1}u^{\frac{\gamma_n}{\beta_n}-1}dY du\cdots dy_2dy_1\\
&=\prod_{q=1}^{n+1}\left(\frac{\alpha_q^{\gamma_q}}{\beta_q}\right)\int_{u=0}^{1}(1-u)^{\frac{\gamma_{n+1}}{\beta_{n+1}}-1}u^{\frac{\gamma_n}{\beta_n}-1}du\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{Y=0}^{\left(1-\sum_{i=1}^{n-1}y_i\right)}F\left(Y+\sum_{i=1}^{n-1}y_i\right)Y^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}-1}\prod_{\lambda=1}^{n-1}\left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dY\cdots dy_2dy_1\\
&=\prod_{q=1}^{n+1}\left(\frac{\alpha_q^{\gamma_q}}{\beta_q}\right)\frac{\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1}\cdots\int_{Y=0}^{\left(1-\sum_{i=1}^{n-1}y_i\right)}F\left(Y+\sum_{i=1}^{n-1}y_i\right)Y^{\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}-1}\prod_{\lambda=1}^{n-1}\left(y_\lambda^{\frac{\gamma_\lambda}{\beta_\lambda}-1}\right)dY\cdots dy_2dy_1\end{aligned}$$

Apply the *n* transformation equations:

$y_m=\left(\frac{z_m}{\alpha_m}\right)^{\beta_m}\ \forall m\le(n-1)\Rightarrow dy_{n-1}\cdots dy_2dy_1=\prod_{m=1}^{n-1}\left[\frac{\beta_m}{\alpha_m}\left(\frac{z_m}{\alpha_m}\right)^{\beta_m-1}\right]dz_{n-1}\cdots dz_2dz_1$ and

$Y=\left(\frac{z_n}{\alpha_n}\right)^{\beta_n}\Rightarrow dY=\frac{\beta_n}{\alpha_n}\left(\frac{z_n}{\alpha_n}\right)^{\beta_n-1}dz_n$

$$\begin{aligned}J_{n+1}&=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\alpha_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\int_{z_1=0}^{\alpha_1 t^{\frac{1}{\beta_1}}}\int_{z_2=0}^{\alpha_2\left[t-\left(\frac{z_1}{\alpha_1}\right)^{\beta_1}\right]^{1/\beta_2}}\cdots\int_{z_{n+1}=0}^{\alpha_{n+1}\left[t-\sum_{k=1}^{n}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\right]^{1/\beta_{n+1}}}F\left[\sum_{i=1}^{n}\left(\frac{z_i}{\alpha_i}\right)^{\beta_i}\right]z_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda-1}\right)dz_n\cdots dz_2dz_1\\
&=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\alpha_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\iint_{V^n}\cdots\int F\left[\sum_{i=1}^{n}\left(\frac{z_i}{\alpha_i}\right)^{\beta_i}\right]z_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\prod_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda-1}\right)dz_n\cdots dz_2dz_1,\text{ which, by hypothesis}\\
&=\frac{\alpha_{n+1}^{\gamma_{n+1}}\Gamma\left(\frac{\gamma_n}{\beta_n}\right)\Gamma\left(\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}{\alpha_n^{\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}\beta_{n+1}\Gamma\left(\frac{\gamma_n}{\beta_n}+\frac{\gamma_{n+1}}{\beta_{n+1}}\right)}\frac{\prod_{k=1}^{n-1}\left[\frac{\alpha_k^{\gamma_k}}{\beta_k}\Gamma\left(\frac{\gamma_k}{\beta_k}\right)\right]\left(\frac{\alpha_n^{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}}{\beta_n}\right)\Gamma\left(\frac{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}{\beta_n}\right)}{\Gamma\left(\sum_{i=1}^{n-1}\frac{\gamma_i}{\beta_i}+\frac{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}{\beta_n}\right)}\int_{u=0}^{t}u^{\sum_{i=1}^{n-1}\left(\frac{\gamma_i}{\beta_i}\right)+\frac{\gamma_n+\frac{\beta_n\gamma_{n+1}}{\beta_{n+1}}}{\beta_n}-1}F(u)\,du\\
&=\frac{\prod_{k=1}^{n+1}\left[\frac{\alpha_k^{\gamma_k}}{\beta_k}\Gamma\left(\frac{\gamma_k}{\beta_k}\right)\right]}{\Gamma\left(\sum_{i=1}^{n+1}\frac{\gamma_i}{\beta_i}\right)}\int_{u=0}^{t}u^{\sum_{i=1}^{n+1}\left(\frac{\gamma_i}{\beta_i}\right)-1}F(u)\,du\text{ the required result.}\end{aligned}$$

Corollary 2.4: If $\alpha_p,\beta_q,\Re[\gamma_r]>0\forall p,q,r$, $V^n:=\left\{(z_1,z_2,\ldots,z_n)\in\mathbb{R}^n\mid z_j\ge 0\forall j,t_1\le\sum_{k=1}^{n}\left(\frac{z_k}{\alpha_k}\right)^{\beta_k}\le t_2\right\}$, and if F is continuous on $[t_1,t_2]$ then $\iint_{V^n}\cdots\int F\left[\sum_{i=1}^{n}\left(\frac{z_i}{\alpha_i}\right)^{\beta_i}\right]\prod_{\lambda=1}^{n}\left(z_\lambda^{\gamma_\lambda-1}\right)dz_n\cdots dz_2dz_1=\frac{\prod_{k=1}^{n}\left[\frac{\alpha_k^{\gamma_k}}{\beta_k}\Gamma\left(\frac{\gamma_k}{\beta_k}\right)\right]}{\Gamma\left(\sum_{i=1}^{n}\frac{\gamma_i}{\beta_i}\right)}\int_{u=t_1}^{t_2}u^{\sum_{i=1}^{n}\left(\frac{\gamma_i}{\beta_i}\right)-1}F(u)\,du$.

Exercises II

2.1) a) Show that $\prod_{q=1}^{n}\left[\frac{\alpha_q^{\gamma_q}}{\beta_q}\Gamma\left(\frac{\gamma_q}{\beta_q}\right)\right]\Big/\Gamma\left(1+\sum_{k=1}^{n}\frac{\gamma_k}{\beta_k}\right)=\frac{\beta_1}{\gamma_1}\prod_{k=1}^{n-1}\mathrm{B}\left(1+\sum_{j=1}^{k}\frac{\gamma_j}{\beta_j},\frac{\gamma_{k+1}}{\beta_{k+1}}\right)$.

b) Rewrite Theorem 2.1 using the result of part a.

c) Formulate an alternate proof of Theorem 2.1 by means of a product of beta functions.

2.2) Prove Corollary 2.1.

2.3)

III. A Bit of Geometry

The orthotope is the generalization of the rectangular parallelepiped to n-space.

DEFINITION OF ORTHOTOPE HERE

Consider the orthotope with polytope vertices of the form $(\pm b_1, \pm b_2, \ldots, \pm b_n)$ orientated with facet-centered axes determined by the set $P^n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid |x_k| \le b_k \forall k\}$. It is convenient in this work to develop an alternate determination of the orthotope by means of the set $Q^n := \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \lim\limits_{N\to\infty} \sum\limits_{k=1}^{n} \left(\frac{x_k}{b_k}\right)^{2N} \le n\right\}$, where $N \to \infty$ through positive integral values. The equivalence of said sets, desired because P^n follows intuitively from the general notion of an orthotope whereas Q^n is of a form more suitable for use in conjunction with Dirichlet integrals, shall be discussed presently.

$$P^n = Q^n \Leftrightarrow \left[\left(P^n \subseteq Q^n\right) \wedge \left(Q^n \subseteq P^n\right)\right] \Leftrightarrow \left[\left(x \in P^n \Rightarrow x \in Q^n\right) \wedge \left(x \in Q^n \Rightarrow x \in P^n\right)\right] \Leftrightarrow \left(x \in P^n \Leftrightarrow x \in Q^n\right)$$

$$\text{Sufficiency: define } y_k := \lim_{N\to\infty}\left(\frac{x_k}{b_k}\right)^{2N} = \begin{cases} 0, & |x_k| < b_k \\ 1, & |x_k| = b_k \\ \infty, & |x_k| > b_k \end{cases} \text{ so that } x \in P^n \Rightarrow 0 \le y_k \le 1 \forall k \Rightarrow 0 \le \sum_{k=1}^{n} y_k \le n \Rightarrow x \in Q^n;$$

$$\text{Necessity: } x \in Q^n \Rightarrow \sum_{k=1}^{n} y_k \le n \Rightarrow |x_k| \le b_k \forall k \Rightarrow x \in P^n.$$

In order that Q^n may be utilized with regards to Dirichlet integrals, consider now the orthotope with polytope edges of lengths $b_1, b_2, \ldots, b_n$ given by the set $U^n := Q^n \cap \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_k \ge 0 \forall k\}$; define the sequence of sets $U_N^n := \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_k \ge 0 \forall k, \sum\limits_{k=1}^{n}\left(\frac{x_k}{b_k}\right)^{2N} \le n\right\}$, so that $\lim\limits_{N\to\infty} U_N^n = U^n$ and one may check that

$$\text{content}(\textit{orthotope}) := \iint_{U^n}\!\cdots\!\int dU^n = \lim_{N\to\infty} \iint_{U_N^n}\!\cdots\!\int dU_N^n = \lim_{N\to\infty} n^{\sum_{p=1}^{n}\frac{1}{2N}} \prod_{q=1}^{n}\left[\frac{b_q}{2N}\Gamma\left(\frac{1}{2N}\right)\right] \Big/ \Gamma\left(1 + \sum_{k=1}^{n}\frac{1}{2N}\right)$$

$$= \prod_{q=1}^{n}(b_q) \lim_{N\to\infty} n^{\frac{n}{2N}} \Gamma^n\left(1 + \frac{1}{2N}\right) \Big/ \Gamma\left(1 + \frac{n}{2N}\right) = \prod_{q=1}^{n}(b_q), \text{ as it must.}$$

The hypercube, formally referred to as the measure polytope, is the generalization of the cube to n-space and moreover the degenerate case of the orthotope with all equal length polytope edges. Thus, upon substituting $b_q = c \forall q$ in the above formula, one arrives at $\text{content}(\textit{hypercube}) = c^n$.

Content of a hyperellipsoid: The hyperellipsoid is the generalization of the ellipsoid to n-space. If the lengths of its semi-axes are $a_1, a_2, \ldots, a_n > 0$ then its equation is $\sum\limits_{q=1}^{n}\left(\frac{x_q}{a_q}\right)^2 = 1$, which suggests the multiple integral be taken over the set $E^n := \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_j \ge 0 \forall j, \sum\limits_{q=1}^{n}\left(\frac{x_q}{a_q}\right)^2 \le 1\right\}$. Each of the n coordinates may either be negative or non-negative, thus by Corollary 2.3 it follows that

$$\text{content}(\textit{hyperellipse}) = \left({}_2\mathrm{C}_1\right)^n \text{content}(E^n) = 2^n \iint_{E^n}\!\cdots\!\int dx_n \cdots dx_2 dx_1 = 2^n \prod_{q=1}^{n}\left[\frac{a_q}{2}\Gamma\left(\frac{1}{2}\right)\right] \Big/ \Gamma\left(1 + \sum_{k=1}^{n}\frac{1}{2}\right) = \frac{\pi^{\frac{n}{2}} \prod_{q=1}^{n} a_q}{\Gamma\left(1 + \frac{n}{2}\right)},$$

where ${}_2\mathrm{C}_1$ is a binomial coefficient.

Content of a hypersphere: The hypersphere is the generalization of the sphere to n-space and moreover the degenerate case of the hyperellipsoid with all equal length semi-axes. Thus, upon substituting $a_q = r \forall q$ in the above formula, one arrives at $\text{content}(\textit{hypersphere}) = \pi^{\frac{n}{2}} r^n \Big/ \Gamma\left(1+\frac{n}{2}\right)$.

Exercises III

3.1)

3.2) Show that the set $R_d^n := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \lim_{N\to\infty} \sum_{k=1}^{n} \left(\frac{x_k}{b_k}\right)^{2N} = n-d \right\}$ possesses that property of describing precisely the d-dimensional content (less lower dimensional boundaries) of an n-dimensional orthotope; that is to say that the sets $R_0^n, R_1^n, \ldots, R_{n-2}^n, R_{n-1}^n, R_n^n$ describe the vertices, edges..., ridges, facets, and hypervolume of an n-dimensional orthotope, respectively. Note that $R_i^n \cap R_j^n = \varnothing \forall i \neq j$ and $\bigcup_{k=0}^{n} R_k^n = U^n$, where $\varnothing$ denotes the null (empty) set and U^n is the orthotope defined above.

3.3)

IV. Multiple Integral Representations of Generalized Hypergeometric Functions

TEXT HERE

It is perhaps best to begin with the well-known result of

Example 4.1: Show that $Z(n) := \int_0^1 \int_0^1 \cdots \int_0^1 \left(1 - \prod_{k=1}^{n} \lambda_k\right)^{-1} d\lambda_n \cdots d\lambda_2 d\lambda_1 = \zeta(n) \; \forall n \in \mathbb{Z}^+$, where $\zeta(\cdot)$ is the Riemann zeta function. The domain of integration is the unit n-dimensional hypercube orientated such that: one of its 2^n vertices is at the origin, the vertex which is furthest from said vertex is at $(1,1,\ldots,1)$, and the set of points defined by it lie entirely in the portion of n-space with positive coordinates. This observation alone suggests use of set $C^n := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_k \geq 0 \forall k, \lim_{N\to\infty} \sum_{q=1}^{n} \lambda_q^{2N} \leq n \right\}$. However, in observance of the integrand's infinite discontinuity at the point, moreover the vertex, $\prod_{k=1}^{n} \lambda_k = 1 \Leftrightarrow \lambda_k = 1 \forall k$, one is careful to define the sequences of sets $C_N^n := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_k \geq 0 \forall k, \sum_{q=1}^{n} \lambda_q^{2N} \leq n \right\}$, so that $\lim_{N\to\infty} C_N^n = C^n$ and one's integral, namely $Z_N(n) := \iint \cdots \int_{C_N^n} \frac{\prod_{q=1}^{n} d\lambda_q}{1 - \prod_{k=1}^{n} \lambda_k}$, is now a proper one. Evaluation is as follows, write

$$Z(n) = \lim_{N\to\infty} Z_N(n) = \lim_{N\to\infty} \iint \cdots \int_{C_N^n} \frac{\prod_{q=1}^{n} d\lambda_q}{1 - \prod_{k=1}^{n} \lambda_k} = \lim_{N\to\infty} \iint \cdots \int_{C_N^n} \sum_{\rho=1}^{\infty} \left[\prod_{k=1}^{n} \left(\lambda_k^{\rho-1} d\lambda_k \right) \right] = \lim_{N\to\infty} \sum_{\rho=1}^{\infty} \left[\iint \cdots \int_{C_N^n} \prod_{k=1}^{n} \left(\lambda_k^{\rho-1} d\lambda_k \right) \right],$$

noting that the geometric series expansion of the integrand does converge to the required values at every point an element of C_N^n *before* the limit as $N \to \infty$ is applied. Then, applying Theorem 1, the integral becomes,

$$Z(n) = \lim_{N\to\infty} \sum_{\rho=1}^{\infty} \left\{ \prod_{q=1}^{n} \left[\frac{1^\rho}{2N} \Gamma\left(\frac{\rho}{2N}\right) \right] \Big/ \Gamma\left(1 + \sum_{k=1}^{n} \frac{\rho}{2N}\right) \right\} = \lim_{N\to\infty} \sum_{\rho=1}^{\infty} \left\{ \left(\frac{1}{2N}\right)^n \left[\Gamma\left(\frac{\rho}{2N}\right) \right]^n \Big/ \Gamma\left(1 + \sum_{k=1}^{n} \frac{\rho}{2N}\right) \right\}$$

$$= \lim_{N\to\infty} \sum_{\rho=1}^{\infty} \left\{ \frac{1}{\rho^n} \left[\Gamma\left(1 + \frac{\rho}{2N}\right) \right]^n \Big/ \Gamma\left(1 + \frac{\rho n}{2N}\right) \right\} = \sum_{\rho=1}^{\infty} \frac{1}{\rho^n} = \zeta(n).$$

An alternative method of evaluation: recall that for $|z|<1$, $\frac{1}{1\text{-}z}=\prod_{q=0}^{\infty}\left(1+z^{2^q}\right)$; let $C_N^n, C^n, Z_N(n)$, and $Z(n)$ be as above and write,

$$Z(n)=\lim_{N\to\infty}Z_N(n)=\lim_{N\to\infty}\iint_{C_N^n}\cdots\int\frac{\prod_{q=1}^{n}d\lambda_q}{1-\prod_{k=1}^{n}\lambda_k}=\lim_{N\to\infty}\iint_{C_N^{n-1}}\cdots\int\left\{\ln\left[\left(1-\prod_{k=1}^{n-1}\lambda_k\right)^{-1}\right]\Big/\prod_{j=1}^{n-1}\lambda_j\right\}\prod_{i=1}^{n-1}d\lambda_i$$

$$=\lim_{N\to\infty}\iint_{C_N^{n-1}}\cdots\int\left\{\ln\left[\prod_{q=0}^{\infty}\left(1+\prod_{k=1}^{n-1}\lambda_k^{2^q}\right)\right]\Big/\prod_{j=1}^{n-1}\lambda_j\right\}\prod_{i=1}^{n-1}d\lambda_i=\lim_{N\to\infty}\iint_{C_N^{n-1}}\cdots\int\left\{\sum_{q=0}^{\infty}\left[\ln\left(1+\prod_{k=1}^{n-1}\lambda_k^{2^q}\right)\right]\Big/\prod_{j=1}^{n-1}\lambda_j\right\}\prod_{i=1}^{n-1}d\lambda_i$$

$$=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\iint_{C_N^{n-1}}\cdots\int\left[\ln\left(1+\prod_{k=1}^{n-1}\lambda_k^{2^q}\right)\Big/\prod_{j=1}^{n-1}\lambda_j\right]\prod_{i=1}^{n-1}d\lambda_i\right\}=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\iint_{C_N^{n-1}}\cdots\int\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j}\prod_{k=1}^{n-1}\left(\lambda_k^{j2^q-1}d\lambda_k\right)\right]\right\}$$

$$=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j}\iint_{C_N^{n-1}}\cdots\int\prod_{k=1}^{n-1}\left(\lambda_k^{j2^q-1}d\lambda_k\right)\right]\right\},\text{ which, by Theorem 1}$$

$$=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j}\prod_{k=1}^{n-1}\left[\frac{1^{j2^q}}{2N}\Gamma\left(\frac{j2^q}{2N}\right)\right]\Big/\Gamma\left(1+\sum_{i=1}^{n-1}\frac{j2^q}{2N}\right)\right]\right\}$$

$$=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j}\left(\frac{1}{2N}\right)^{n-1}\left[\Gamma\left(\frac{j2^q}{2N}\right)\right]^{n-1}\Big/\Gamma\left(1+\frac{j2^q(n-1)}{2N}\right)\right]\right\}$$

$$=\lim_{N\to\infty}\sum_{q=0}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j}\left(\frac{1}{j2^q}\right)^{n-1}\left[\Gamma\left(1+\frac{j2^q}{2N}\right)\right]^{n-1}\Big/\Gamma\left(1+\frac{j2^q(n-1)}{2N}\right)\right]\right\}$$

$$=\sum_{q=0}^{\infty}\left\{\left(2^{1-n}\right)^q\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j^n}\right]\right\}=\left(1-2^{1-n}\right)^{-1}\sum_{j=1}^{\infty}\left[\frac{(-1)^{j+1}}{j^n}\right]=\zeta(n).$$

TEXT HERE

Definition 4.1: The generalized hypergeometric function of a single variable

$${}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix};z\right):=\sum_{\lambda=0}^{\infty}\left\{\frac{z^\lambda}{\lambda!}\prod_{k=1}^{p}\left[\frac{\Gamma(a_k+\lambda)}{\Gamma(a_k)}\right]\prod_{j=1}^{q}\left[\frac{\Gamma(b_j)}{\Gamma(b_j+\lambda)}\right]\right\}$$

Theorem 4.1:

$${}_{n+1}F_n\left(\begin{matrix}a_1,a_2,\ldots,a_n,a_{n+1}\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_{k+1})\Gamma(b_k-a_{k+1})}\right]\iint_{C^n}\cdots\int\prod_{q=1}^{n}\left[t_q^{a_{q+1}-1}\left(1-t_q\right)^{b_q-a_{q+1}-1}\right]\left(1-z\prod_{\lambda=1}^{n}t_\lambda\right)^{-a_1}dt_1\cdots dt_{n-1}dt_n$$

Proof:

Set $K:=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_{k+1})\Gamma(b_k-a_{k+1})}\right]$. Let Ω_n denote the integral of Theorem 4.2 that one may write

$$\Omega_n:=K\iint_{C^n}\cdots\int\prod_{q=1}^{n}\left[t_q^{a_{q+1}-1}\left(1-t_q\right)^{b_q-a_{q+1}-1}\right]\left(1-z\prod_{\lambda=1}^{n}t_\lambda\right)^{-a_1}dt_1\cdots dt_{n-1}dt_n$$

$$=K\sum_{\lambda=0}^{\infty}\frac{\Gamma(a_1+\lambda)z^\lambda}{\Gamma(a_1)\lambda!}\iint_{C^n}\cdots\int\prod_{q=1}^{n}\left[t_q^{a_{q+1}+\lambda-1}\left(1-t_q\right)^{b_q-a_{q+1}-1}\right]dt_1\cdots dt_{n-1}dt_n$$

$$=K\sum_{\lambda=0}^{\infty}\frac{\Gamma(a_1+\lambda)z^\lambda}{\Gamma(a_1)\lambda!}\prod_{q=1}^{n}\mathrm{B}\left(a_{q+1}+\lambda,b_q-a_{q+1}\right)$$

$$=\sum_{\lambda=0}^{\infty}\frac{\Gamma(a_n+\lambda)z^\lambda}{\Gamma(a_n)\lambda!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+\lambda)\Gamma(b_q)}{\Gamma(a_q)\Gamma(b_q+\lambda)}\right]=:{}_{n+1}F_n\left(\begin{matrix}a_1,a_2,\ldots,a_n,a_{n+1}\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)$$

Corollary 4.1:

Lerch Transcendent: $\Phi(z,n,y):=\sum_{q=0}^{\infty}\frac{z^q}{(q+y)^n}=\iint_{C^n}\cdots\int\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Legendre Chi Function: $\chi_n(z):=\sum_{q=0}^{\infty}\frac{z^{2q+1}}{(2q+1)^n}=z\iint_{C^n}\cdots\int\left(1-z^2\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Polygamma Function: $\psi_n(z)=\sum_{q=0}^{\infty}\frac{(-1)^{n+1}n!}{(z+q)^{n+1}}=(-1)^{n+1}n!\iint_{C^{n+1}}\cdots\int\prod_{k=1}^{n+1}\left(\lambda_k^{z-1}\right)\left(1-\prod_{q=1}^{n+1}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Polylogarithm of Order n: $\mathrm{Li}_n(z):=\sum_{q=1}^{\infty}\frac{z^q}{q^n}=z\iint_{C^n}\cdots\int\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Hurwitz Zeta Function: $\zeta(n,y):=\sum_{q=0}^{\infty}\frac{1}{(q+y)^n}=\iint_{C^n}\cdots\int\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Riemann Zeta Function: $\zeta(n):=\sum_{q=1}^{\infty}\frac{1}{q^n}=\iint_{C^n}\cdots\int\left(1-\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Beta Function: $\beta(n):=\sum_{q=0}^{\infty}\frac{(-1)^q}{(2q+1)^n}=\iint_{C^n}\cdots\int\left(1+\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Eta Function: $\eta(n):=\sum_{q=1}^{\infty}\frac{(-1)^{q-1}}{q^n}=\iint_{C^n}\cdots\int\left(1+\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Lambda Function: $\lambda(n):=\sum_{q=0}^{\infty}\frac{1}{(2q+1)^n}=\iint_{C^n}\cdots\int\left(1-\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Theorem 4.2:

$$ {}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\cdots\int\exp\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}\left(1-t_q\right)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n $$

Proof:

Set $K':=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]$. Let Λ_n denote the integral of Theorem 4.2 that one may write

$$ \begin{aligned}\Lambda_n&:=K'\iint_{C^n}\cdots\int\exp\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}\left(1-t_q\right)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n\\ &=K'\sum_{\lambda=0}^{\infty}\frac{z^\lambda}{\lambda!}\iint_{C^n}\cdots\int\prod_{q=1}^{n}\left[t_q^{a_q+\lambda-1}\left(1-t_q\right)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n\\ &=K'\sum_{\lambda=0}^{\infty}\frac{z^\lambda}{\lambda!}\prod_{q=1}^{n}\mathrm{B}\left(a_q+\lambda,b_q-a_q\right)\\ &=\sum_{\lambda=0}^{\infty}\frac{z^\lambda}{\lambda!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+\lambda)\Gamma(b_q)}{\Gamma(a_q)\Gamma(b_q+\lambda)}\right]=:{}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)\end{aligned} $$

Exercises IV

4.1) Prove Gauss's Hypergeometric Theorem: ${}_2F_1\left(\begin{matrix}a_1,a_2\\ b_1\end{matrix};1\right)=\frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}$.

4.2) Use the Theorem 4.1 to prove:

$$ \frac{d^m}{dz^m}\left[{}_{n+1}F_n\left(\begin{matrix}a_1,\ldots,a_n,a_{n+1}\\ b_1,\ldots,b_n\end{matrix};z\right)\right]=\frac{\Gamma(a_{n+1}+m)}{\Gamma(a_{n+1})}\prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_\lambda+m)\Gamma(b_\lambda)}{\Gamma(a_\lambda)\Gamma(b_\lambda+m)}\right]{}_{n+1}F_n\left(\begin{matrix}a_1+m,\ldots,a_n+m,a_{n+1}+m\\ b_1+m,\ldots,b_n+m\end{matrix};z\right) $$

4.3) Use the Theorem 4.2 to prove:

$$ \frac{d^m}{dz^m}\left[{}_nF_n\left(\begin{matrix}a_1,\ldots,a_n\\ b_1,\ldots,b_n\end{matrix};z\right)\right]=\prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_\lambda+m)\Gamma(b_\lambda)}{\Gamma(a_\lambda)\Gamma(b_\lambda+m)}\right]{}_nF_n\left(\begin{matrix}a_1+m,\ldots,a_n+m\\ b_1+m,\ldots,b_n+m\end{matrix};z\right) $$

4.4) Show that

a) $$\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\!\cdots\!\int \cosh\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n=\sum_{\lambda=0}^{\infty}\frac{z^{2\lambda}}{(2\lambda)!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+2\lambda)\Gamma(b_q)}{\Gamma(a_k)\Gamma(b_q+2\lambda)}\right]$$

b) $$\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\!\cdots\!\int \sinh\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n=\sum_{\lambda=0}^{\infty}\frac{z^{2\lambda+1}}{(2\lambda+1)!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+2\lambda+1)\Gamma(b_q)}{\Gamma(a_k)\Gamma(b_q+2\lambda+1)}\right]$$

Hint: consider the quantity $\frac{1}{2}\left[{}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)\pm{}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};-z\right)\right]$.

4.5) Use the result of Exercise 4.4 to show that

a) $$\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\!\cdots\!\int \cos\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n=\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda z^{2\lambda}}{(2\lambda)!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+2\lambda)\Gamma(b_q)}{\Gamma(a_k)\Gamma(b_q+2\lambda)}\right]$$

b) $$\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\!\cdots\!\int \sin\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n=\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda z^{2\lambda+1}}{(2\lambda+1)!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+2\lambda+1)\Gamma(b_q)}{\Gamma(a_k)\Gamma(b_q+2\lambda+1)}\right]$$

4.6) Show that

Answers to Selected Exercises

I.

1.1) $$\Gamma(x+1):=\lim_{\lambda\to\infty}\frac{\lambda!\lambda^x}{(x+1)(x+2)\cdots(x+\lambda)}=\underbrace{\lim_{\lambda\to\infty}\frac{\lambda x}{x+\lambda}}_{=x}\cdot\lim_{\lambda\to\infty}\frac{\lambda!\lambda^{x-1}}{x(x+1)(x+2)\cdots(x+\lambda-1)}=x\,\Gamma(x)$$

1.2) a) The principle of induction affords that if one can demonstrate both (i) the truth of P_1 and (ii) $P_n\Rightarrow P_{n+1}$, then the truth of the infinite sequence of propositions $\{P_n\}_{n=1}^{\infty}$ is established.

(i) $$\Gamma(2)=\Gamma(1+1)=1\cdot\Gamma(1)=\lim_{\lambda\to\infty}\left.\frac{\lambda!\lambda^{x-1}}{x(x+1)\cdots(x+\lambda-1)}\right|_{x=1}=\lim_{\lambda\to\infty}\frac{\lambda!\lambda^0}{\lambda!}=1=1!.$$

(ii) Assume that $\forall n\in\mathbb{Z}^*$, $\Gamma(n+1)=n!$, then $\Gamma(n+2)=(n+1)\Gamma(n+1)=(n+1)n!=(n+1)!$.

b) Substituting $n=0$ in Equation 1.2 yields $0!=\Gamma(1)$, which is unity.

1.3)

$$\lim_{N\to\infty}\frac{\Gamma^n\left(1+\frac{k}{N}\right)}{\Gamma\left(1+\frac{kn}{N}\right)}=\lim_{N\to\infty}\frac{\left(1+\frac{k}{N}\right)^{-n}\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^{n\left(1+\frac{k}{N}\right)}\Big/\left(1+\frac{1+\frac{k}{N}}{\rho}\right)^{n}\right]}{\left(1+\frac{kn}{N}\right)^{-1}\prod_{j=1}^{\infty}\left[\left(1+\frac{1}{j}\right)^{1+\frac{kn}{N}}\Big/\left(1+\frac{1+\frac{kn}{N}}{j}\right)\right]}$$

$$=\lim_{N\to\infty}\left(1+\frac{k}{N}\right)^{-n}\left(1+\frac{kn}{N}\right)\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^{n-1}\left(1+\frac{1+\frac{kn}{N}}{\rho}\right)\left(1+\frac{1+\frac{k}{N}}{\rho}\right)^{-n}\right]$$

$$=\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^{n-1}\lim_{N\to\infty}\left(1+\frac{1+\frac{kn}{N}}{\rho}\right)\left(1+\frac{1+\frac{k}{N}}{\rho}\right)^{-n}\right]=\prod_{\rho=1}^{\infty}\left[\left(1+\frac{1}{\rho}\right)^{n-1}\left(1+\frac{1}{\rho}\right)\left(1+\frac{1}{\rho}\right)^{-n}\right]=1.$$

1.4)

1.5)

1.6) $\int_{z=\frac{1}{2}}^{1} \psi_1(z)\,dz = \psi_0(z)\Big|_{z=\frac{1}{2}}^{1} = \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \textbf{INSERT WORK HERE} = 2\ln 2$

II.

2.1) Take $\alpha_k = \alpha_k^{'} t^{\frac{1}{\beta_k}}$ in Theorem 2.1.

2.2)

2.3)

III.

3.1)

3.2)

3.3)

IV.

4.1) $_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; 1\right) = \frac{\Gamma(b_1)}{\Gamma(a_2)\Gamma(b_1 - a_2)} \int_{t_1=0}^{1} t_1^{a_2-1}(1-t_1)^{b_1-a_1-a_2-1}\,dt_1 = \frac{\Gamma(b_1)\mathrm{B}(a_2, b_1-a_1-a_2)}{\Gamma(a_2)\Gamma(b_1-a_2)} = \frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}$

4.2) Let K be as in the proof of Theorem 4.1 and write

$$
\begin{aligned}
\frac{d^m}{dz^m}\left[{}_{n+1}F_n\left(\begin{matrix} a_1,\ldots,a_{n+1} \\ b_1,\ldots,b_n \end{matrix}; z\right)\right] &= \frac{d^m}{dz^m}\left\{K \iint_{C^n}\cdots\int \prod_{q=1}^{n}\left[t_q^{a_{q+1}-1}(1-t_q)^{b_q-a_{q+1}-1}\right]\left(1 - z\prod_{\lambda=1}^{n} t_\lambda\right)^{-a_1} dt_1\cdots dt_{n-1}dt_n\right\} \\
&= K \iint_{C^n}\cdots\int \prod_{q=1}^{n}\left[t_q^{a_{q+1}-1}(1-t_q)^{b_q-a_{q+1}-1}\right]\frac{d^m}{dz^m}\left[\left(1 - z\prod_{\lambda=1}^{n} t_\lambda\right)^{-a_1}\right] dt_1\cdots dt_{n-1}dt_n \\
&= K\frac{\Gamma(a_1+m)}{\Gamma(a_1)} \iint_{C^n}\cdots\int \prod_{q=1}^{n}\left[t_q^{a_{q+1}+m-1}(1-t_q)^{b_q-a_{q+1}-1}\right]\left(1 - z\prod_{\lambda=1}^{n} t_\lambda\right)^{-(a_1+m)} dt_1\cdots dt_{n-1}dt_n \\
&= K\frac{\Gamma(a_1+m)}{\Gamma(a_1)}\prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_{\lambda+1}+m)\Gamma(b_\lambda - a_{\lambda+1})}{\Gamma(b_\lambda+m)}\right]{}_{n+1}F_n\left(\begin{matrix} a_1+m,\ldots,a_{n+1}+m \\ b_1+m,\ldots,b_n+m \end{matrix}; z\right) \\
&= \frac{\Gamma(a_{n+1}+m)}{\Gamma(a_{n+1})}\prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_\lambda+m)\Gamma(b_\lambda)}{\Gamma(a_\lambda)\Gamma(b_\lambda+m)}\right]{}_{n+1}F_n\left(\begin{matrix} a_1+m,\ldots,a_{n+1}+m \\ b_1+m,\ldots,b_n+m \end{matrix}; z\right)
\end{aligned}
$$

4.3) Let K' be as in the proof of Theorem 4.2 and write

$$
\begin{aligned}
\frac{d^m}{dz^m}\left[{}_nF_n\left(\begin{matrix} a_1,a_2,\ldots,a_n \\ b_1,b_2,\ldots,b_n \end{matrix}; z\right)\right] &= \frac{d^m}{dz^m}\left\{K' \iint_{C^n}\cdots\int \exp\left(z\prod_{\lambda=1}^{n} t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n\right\} \\
&= K' \iint_{C^n}\cdots\int \frac{d^m}{dz^m}\left[\exp\left(z\prod_{\lambda=1}^{n} t_\lambda\right)\right]\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n \\
&= K' \iint_{C^n}\cdots\int \exp\left(z\prod_{\lambda=1}^{n} t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q+m-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n \\
&= K'\prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_\lambda+m)\Gamma(b_\lambda-a_\lambda)}{\Gamma(b_\lambda+m)}\right]{}_nF_n\left(\begin{matrix} a_1+m,\ldots,a_n+m \\ b_1+m,\ldots,b_n+m \end{matrix}; z\right) \\
&= \prod_{\lambda=1}^{n}\left[\frac{\Gamma(a_\lambda+m)\Gamma(b_\lambda)}{\Gamma(a_\lambda)\Gamma(b_\lambda+m)}\right]{}_nF_n\left(\begin{matrix} a_1+m,\ldots,a_n+m \\ b_1+m,\ldots,b_n+m \end{matrix}; z\right)
\end{aligned}
$$

4.4) a) NOTE: EvenPartOf(nFn) = [nFn(~;z) + nFn(~;-z)]/2

$$\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\cdots\int \cosh\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}\left(1-t_q\right)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n=\sum_{\lambda=0}^{\infty}\frac{z^{2\lambda}}{(2\lambda)!}\prod_{q=1}^{n}\left[\frac{\Gamma(a_q+2\lambda)\Gamma(b_q)}{\Gamma(a_k)\Gamma(b_q+2\lambda)}\right]$$

$$\frac{1}{2}\left[{}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)+{}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};-z\right)\right]$$

b) NOTE: OddPartOf(nFn) = [nFn(~;z) - nFn(~;-z)]/2

4.5)

Quotations To Be Perhaps Added To This Paper:

Quote 3.0: *"Whereas Nature does not admit of more than three dimensions ... it may justly seem very improper to talk of a solid ... drawn into a fourth, fifth, sixth, or further dimension.' --John Wallis*
-In mathematics the art of proposing a question must be held of higher value than solving it. --Georg Cantor, A thesis defended at Cantor's doctoral examination
-The third book [of Conics] contains many remarkable theorems useful for the synthesis of solid loci ... the most and prettiest of these theorems are new, and it was their discovery which made me aware that Euclid did not work out the synthesis of the locus with respect to three and four lines ... for it was not possible for the said synthesis to be completed without the aid of the additional theorems discovered by me. --Apollonius
-(Before section on conjectures): I have had my results for a long time: but I do not yet know how I am to arrive at them. --Carl Friedrich Gauss
-The total number of Dirichlet's publications is not large: jewels are not weighed on a grocery scale. -- Carl Friedrich Gauss

References:

[a] http://numbers.computation.free.fr/Constants/Miscellaneous/gammaFunction.html

[b] Whittaker, E.T., and G.N. Watson. A Course of Modern Analysis (American Ed.). New York: The Macmillian Co., 1944.