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For $n \in \mathbb{Z}^+$, $y \notin \mathbb{N}$, and $z \in \mathbb{C}$ such that $|z| \le R < 1$ for some $R \in (0,1)$, the Lerch Transcendent is given by

$$\begin{aligned}\Phi(z,n,y) &:= \sum_{q=0}^{\infty} \frac{z^q}{(q+y)^n} \\ &= \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1} d\lambda_1\cdots d\lambda_{n-1}d\lambda_n \\ &= \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\frac{\lambda_1^{y-1}\cdots\lambda_{n-1}^{y-1}\lambda_n^{y-1}}{1-z\lambda_1\cdots\lambda_{n-1}\lambda_n} d\lambda_1\cdots d\lambda_{n-1}d\lambda_n\end{aligned}$$

Proof: Let $\varphi(z,n,y)$ denote the above integral, viz.

$$\varphi(z,n,y) := \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1} d\lambda_1\cdots d\lambda_{n-1}d\lambda_n .$$

Expanding $\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1}$ as a geometric series yields

$$\begin{aligned}\varphi(z,n,y) &= \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\sum_{j=0}^{\infty}\left(z^j\prod_{q=1}^{n}\lambda_q^j\right)d\lambda_1\cdots d\lambda_{n-1}d\lambda_n \\ &= \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\sum_{j=0}^{\infty}\left(z^j\prod_{q=1}^{n}\lambda_q^{j+y-1}\right)d\lambda_1\cdots d\lambda_{n-1}d\lambda_n\end{aligned}$$

where the convergence of the geometric series is uniform (since $|z| \le R < 1$). The interchange of the order of integration and summation is therefore justified, giving

$$\begin{aligned}\varphi(z,n,y) &= \sum_{j=0}^{\infty} z^j \int_{\lambda_n=0}^{1}\int_{\lambda_{n-1}=0}^{1}\cdots\int_{\lambda_1=0}^{1}\prod_{q=1}^{n}\left(\lambda_q^{j+y-1}\right)d\lambda_1\cdots d\lambda_{n-1}d\lambda_n \\ &= \sum_{j=0}^{\infty} z^j \prod_{q=1}^{n}\int_{\lambda_q=0}^{1}\lambda_q^{j+y-1}d\lambda_q \\ &= \sum_{j=0}^{\infty} z^j \prod_{q=1}^{n}(j+y)^{-1} \\ &= \sum_{j=0}^{\infty}\frac{z^j}{(j+y)^n} = \Phi(z,n,y)\end{aligned}$$

which is the required result.