and so (7.11) can be written

$$-i\mathfrak{M} = \left(ie\bar{u}_f\gamma^\mu u_i\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(-ij^\nu(\mathbf{q})\right). \tag{7.13}$$

We recognize the familiar vertex factor and photon propagator of the Feynman rules for the amplitude $(-i\mathfrak{M})$, see Section 6.17. We therefore deduce that the factor $-ij^\nu$ is associated with the source. For a static nucleus of charge Ze,

$$\begin{aligned} j^0(\mathbf{x}) &= \rho(\mathbf{x}) = Ze\,\delta(\mathbf{x}) \\ \mathbf{j}(\mathbf{x}) &= 0, \end{aligned} \tag{7.14}$$

and so

$$-i\mathfrak{M} = \left(ie\bar{u}_f\gamma_0 u_i\right)\left(\frac{-i}{q^2}\right)\left(-iZe\right). \tag{7.15}$$

The result is recorded in Fig. 7.1b. For a static nucleus, (7.15) just describes Rutherford scattering. We recognize the familiar result for the angular distribution [see (1.6)]:

$$\frac{d\sigma}{d\Omega} \sim |\mathfrak{M}|^2 \sim \frac{1}{\sin^4(\theta/2)}, \tag{7.16}$$

where θ is the deflection angle of the electron, shown in Fig. 7.2. This angular distribution is a result of the q^{-4} behavior of $d\sigma/d\Omega$ obtained by inserting (7.15) into (7.16). Indeed,

$$\begin{aligned} q^2 &= (p_i - p_f)^2 \\ &\simeq -2k^2(1-\cos\theta) \\ &\simeq -4k^2\sin^2\frac{\theta}{2}, \end{aligned} \tag{7.17}$$

where we have neglected the electron mass and used

$$k \equiv |\mathbf{p}_i| = |\mathbf{p}_f|.$$

7.2 Higher-Order Corrections

The previous calculation gives the Rutherford cross section to $O(\alpha^2)$ and is therefore an approximate perturbative result. To $O(\alpha^4)$, other Feynman diagrams have to be included, one of which is shown in Fig. 7.3. When the invariant

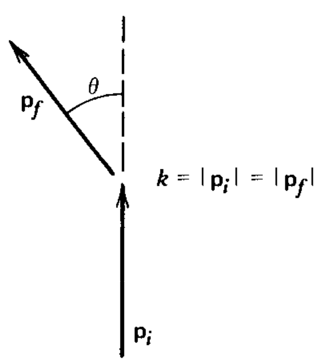

Fig. 7.2 Deflection of an electron by a static charge.

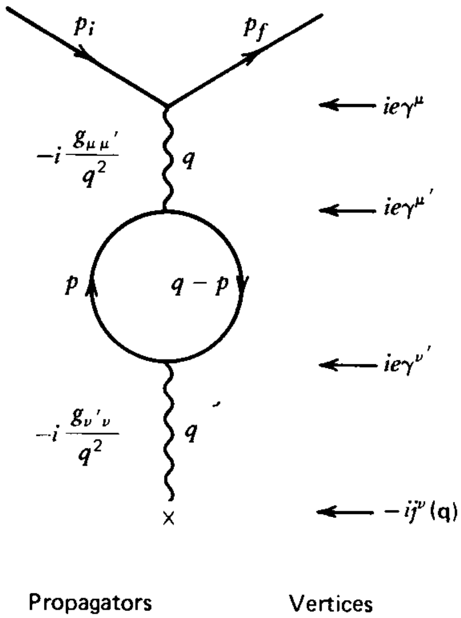

Fig. 7.3 Feynman diagram for Rutherford scattering in which the exchanged photon fluctuates into an e^-e^+ pair.

amplitude $(-i\mathfrak{M})$ of the $O(\alpha^4)$ diagrams is added to (7.15), a more accurate result is obtained for $d\sigma/d\Omega$. In the specific diagram of Fig. 7.3, the exchanged photon spends some time as a virtual e^-e^+ pair; this will lead to a modification of Coulomb's law which results from the lowest-order diagram of Fig. 7.1. We first evaluate the higher-order diagram and then return to discuss this intriguing statement.

By applying the Feynman rules, shown in Fig. 7.3 and Section 6.17, we obtain

$$\begin{aligned}-i\mathfrak{M} = {} & (-1)^1\left(ie\bar{u}_f\gamma^\mu u_i\right)\left(-i\frac{g_{\mu\mu'}}{q^2}\right)\\ & \times\int\frac{d^4p}{(2\pi)^4}\left[\left(ie\gamma^{\mu'}\right)_{\alpha\beta}\frac{i(\not{p}+m)_{\beta\lambda}}{p^2-m^2}\left(ie\gamma^{\nu'}\right)_{\lambda\tau}\frac{i(\not{q}-\not{p}+m)_{\tau\alpha}}{(q-p)^2-m^2}\right]\\ & \times\left(-i\frac{g_{\nu'\nu}}{q^2}\right)\left(-ij^\nu(\mathbf{q})\right).\end{aligned} \tag{7.18}$$

The Feynman rules for higher-order diagrams involve some nontrivial extensions of the rules developed in Chapter 6. A factor $(-1)^n$ should be included in an amplitude for a diagram containing n fermion loops [see, for example, the discussion of Mandl (1959) of the Dyson–Wick formalism], hence the factor $(-1)^1$ in (7.18). The only other unfamiliar feature of (7.18) is the $d^4p/(2\pi)^4$ integration. Its origin is easily uncovered. Although four-momentum is conserved at each vertex, the momentum p, which is circulating around the loop, is unrestricted. The magnitude of the loop four-momentum

$$|p| = \left(p^2\right)^{1/2} = \left(p_0^2 - |\mathbf{p}|^2\right)^{1/2}$$

can be zero or infinite or have any value in between. As p is not observable, we have to sum over all possibilities, hence $\int d^4p$.

The addition of (7.18) to (7.13) can be regarded as a modification to the propagator of the lowest-order result (7.13), namely,

$$-i\frac{g_{\mu\nu}}{q^2} \to -i\frac{g_{\mu\nu}}{q^2} + \left(-i\frac{g_{\mu\mu'}}{q^2}\right)I^{\mu'\nu'}\left(-i\frac{g_{\nu'\nu}}{q^2}\right)$$
$$\to -i\frac{g_{\mu\nu}}{q^2} + \frac{(-i)}{q^2}I_{\mu\nu}\frac{(-i)}{q^2} \tag{7.19}$$

where

$$I_{\mu\nu}(q^2) = (-1)^1\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{(ie\gamma_\mu)\frac{i(\not p + m)}{p^2 - m^2}(ie\gamma_\nu)\frac{i(\not q - \not p + m)}{(q-p)^2 - m^2}\right\}. \tag{7.20}$$

This $O(\alpha)$ modification to the propagator is shown symbolically in Fig. 7.4. The correction can be calculated once and for all and then substituted into any Feynman diagram.

There is, however, a major problem. $I_{\mu\nu}$ as given by (7.20) apparently has terms of the form $\int |p|^3 d|p|/|p|^2$ for $|p| \to \infty$, and so the correction diverges. Indeed, a rather lengthy but straightforward calculation shows that $I_{\mu\nu}$ can be written as

$$I_{\mu\nu} = -ig_{\mu\nu}q^2\, I(q^2) + \cdots \tag{7.21}$$

with

$$I(q^2) = \frac{\alpha}{3\pi}\int_{m^2}^{\infty}\frac{dp^2}{p^2} - \frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\log\left(1 - \frac{q^2 z(1-z)}{m^2}\right), \tag{7.22}$$

where m is the mass of the electron. The dots in (7.21) represent omitted terms which are proportional to $q_\mu q_\nu$ and vanish when the propagator is coupled to external charges or currents. Equation (7.22) divides $I(q^2)$ into logarithmically divergent and finite contributions. We might have expected $I(q^2)$ to diverge quadratically as $\int |p|d|p|$. However, the divergence turns out to be only logarithmic on account of the "conspiratorial" algebra connected with the rest of the integrand. An explicit derivation of (7.21) and (7.22) is given, for example, in Bjorken and Drell (1964), Jauch and Rohrlich (1976), Scadron (1979), or Sakurai (1967) who, in an appendix, gives a collection of tricks for handling loop integrals.

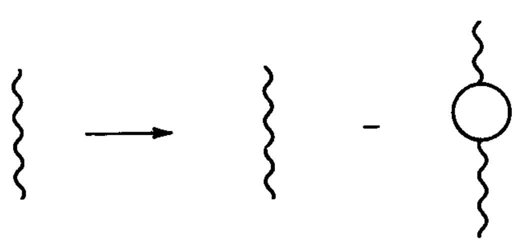

Fig. 7.4 Pictorial representation of (7.19).

Later, we shall study the effects of e^-e^+ loops in the limit of short- or long-range interactions, and so it is useful to evaluate $I(q^2)$ for both large and small values of $(-q^2)$. For $(-q^2)$ small,

$$\log\left(1 - \frac{q^2 z(1-z)}{m^2}\right) \simeq -\frac{q^2 z(1-z)}{m^2},$$

and (7.22) becomes

$$I(q^2) \simeq \frac{\alpha}{3\pi}\log\frac{M^2}{m^2} + \frac{\alpha}{15\pi}\frac{q^2}{m^2}, \tag{7.23}$$

where for the moment we have introduced a cut-off M^2 to replace ∞ as the upper limit of integration in the first term of (7.22). On the other hand, for $(-q^2)$ large,

$$\log\left(1 - \frac{q^2 z(1-z)}{m^2}\right) \simeq \log\left(\frac{-q^2}{m^2}\right)$$

and so, similarly,

$$\begin{aligned} I(q^2) &\simeq \frac{\alpha}{3\pi}\log\left(\frac{M^2}{m^2}\right) - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{m^2}\right) \\ &= \frac{\alpha}{3\pi}\log\left(\frac{M^2}{-q^2}\right). \end{aligned} \tag{7.24}$$

Unless we can dispose of the infinite part of $I(q^2)$ [which appears as $M^2 \to \infty$ in (7.23) and (7.24)], the result will not be physically meaningful.

The way to proceed is best explained by returning to Rutherford scattering. Including the loop contribution, (7.19), the amplitude (7.15) is

$$-i\mathfrak{M} = (ie\bar{u}\gamma_0 u)\left(-\frac{i}{q^2}\right)\left(1 - \frac{\alpha}{3\pi}\log\left(\frac{M^2}{m^2}\right) - \frac{\alpha}{15\pi}\frac{q^2}{m^2} + O(e^4)\right)(-iZe), \tag{7.25}$$

where we have used (7.21) in the small $-q^2$ limit, (7.23), for $I(q^2)$. We may rewrite (7.25) in the form

$$-i\mathfrak{M} = (ie_R\bar{u}\gamma_0 u)\left(-\frac{i}{q^2}\right)\left(1 - \frac{e_R^2}{60\pi^2}\frac{q^2}{m^2}\right)(-iZe_R), \tag{7.26}$$

with

$$\boxed{e_R \equiv e\left(1 - \frac{e^2}{12\pi^2}\log\frac{M^2}{m^2}\right)^{1/2}.} \tag{7.27}$$

To $O(e^4)$, it is easy to verify that (7.25) and (7.26) are mathematically equivalent. In the previous chapters, we were led to believe that e, the charge appearing in the lowest-order Feynman diagrams, is the charge of the electron as measured in

Thomson scattering or any other long-range Coulomb experiment. We never justified this, and it is, in fact, not true! Suppose that e_R in (7.27) is the electric charge listed in the particle data tables, that is, $e_R^2/4\pi = 1/137$. The invariant amplitude (7.26) is now finite. The infinity associated with the cut-off $M \to \infty$ has been "absorbed" in e_R. This procedure is admittedly bizarre. It is our first contact with "renormalization." We return to this later, but first we explore the physics content of our new amplitude, which we have contrived to be free of infinities.

7.3 The Lamb Shift

As pointed out in Chapters 3 and 4, T_{fi} (or $\mathfrak{M}$) represents the Fourier transform of the potential. The first term in (7.26), which is proportional to $|\mathbf{q}|^{-2}$, is associated with the Coulomb potential, since

$$V_0(r) = -\frac{Ze_R^2}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{|\mathbf{q}|^2} = -\frac{Ze_R^2}{4\pi r} \tag{7.28}$$

(see Exercise 6.17). The second term, which represents the quantum effect of the virtual e^-e^+ loop in the propagator of the exchanged photon, contains an extra factor $|\mathbf{q}|^2$ relative to the first. In coordinate space $|\mathbf{q}|^2 \to \nabla^2$ and, since

$$\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} = \delta(\mathbf{r}), \tag{7.29}$$

(7.26) corresponds to an interaction between the electron and the charge Ze_R of the form

$$V(r) = -\left(1 - \frac{e_R^2}{60\pi^2 m^2}\nabla^2\right)\frac{Ze_R^2}{4\pi r}$$

$$\boxed{V(r) = -\frac{Ze_R^2}{4\pi r} - \frac{Ze_R^4}{60\pi^2 m^2}\delta(\mathbf{r}).} \tag{7.30}$$

The extra interaction (including its sign) was anticipated in the discussion of screening in Chapter 1. When $q^2 \to 0$, the electron probes the static charge Ze_R from a large distance and just interacts via the Coulomb interaction, that is, the first term in (7.30). The charge e_R is by definition the familiar electron charge, the one measured in any long-range electromagnetic interaction, for example, Thomson scattering (see Fig. 1.7). But when the electron comes closer to the nucleus (i.e., $-q^2$ increases), it penetrates the cloud of virtual e^-e^+ pairs which surround it. This leads to an increase in the effective interaction as explained in Fig. 1.6 (note, indeed, that both terms in (7.30) have the same sign), and the second term in (7.30) represents a calculation of this effect to leading order or to "the one-loop level." The presence of the loop thus leads to an additional attractive force between the electron and the nucleus.