British Mathematical Olympiad

Round 2 : Thursday, 25 February 1999

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Bucharest, Romania, 13-22 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session (3-7 July) in Birmingham before leaving for Bucharest.

Do not turn over until **told to do so.**

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1. For each positive integer n, let S_n denote the set consisting of the first n natural numbers, that is
$$S_n = \{1, 2, 3, 4, \dots, n-1, n\}.$$
   (i) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sums?

   (ii) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sums?

2. Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA respectively. Prove that PY, QZ, RX are concurrent.

3. Non-negative real numbers p, q and r satisfy $p + q + r = 1$. Prove that
$$7(pq + qr + rp) \leq 2 + 9pqr.$$

4. Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.

   (i) How small can the sum of the digits (in base 10) of such a number be?

   (ii) Can such a number have the sum of its digits (in base 10) equal to 1999?