VII. Symmetries and Quantum Mechanics

Wigner's Theorem

One of the most powerful techniques for extracting consequences about a physical system is to exploit the various symmetries exhibited by the system. A physical system exhibits a symmetry if there is something you can do to the system and after you're finished doing it, the system looks the same as before you did it. Often times, one is able to obtain valuable insights into an appropriate structure of a model Hamiltonian and/or the form of a solution through the symmetries of the physical system being modeled. In addition, the presence of symmetries can sometimes be used to exclude certain transitions leading to selection rules.

The postulates of quantum mechanics place very severe restrictions on the nature of the possible operators which can represent the symmetry transformations. These restrictions are embodied in Wigner's theorem.

Consider two descriptions of the same physical system which are related by the symmetry transformation. The fundamental property of a symmetry is that the descriptions must produce the same physics. Thus, if the states of the system are labeled in the two descriptions by primed and unprimed vectors, then the spectral decomposition postulate and the basic symmetry requirement dictate that

$$\frac{|<\psi'|\phi'>|^2}{<\psi'|\psi'><\phi'|\phi'>} = \frac{|<\psi|\phi>|^2}{<\psi|\psi><\phi|\phi>} . \tag{1}$$

The states in the two descriptions are related to each other by an operator U so that

$$|\phi'>= U|\phi>\equiv |U\phi> \ . \tag{2}$$

However, this does not completely define the operator U since we can always rescale U by a complex number Z_ϕ and still obtain the same physical state, ie. $U|\phi>$ and $Z_\phi U|\phi>$ represent the same physical state (same ray in the Hilbert space). What Wigner showed was that if U is an operator satisfying

$$\frac{|<U\psi|U\phi>|^2}{<U\psi|U\psi><U\phi|U\phi>} = \frac{|<\psi|\phi>|^2}{<\psi|\psi><\phi|\phi>} , \tag{3}$$

then one can always adjust the phases so that U is either a unitary operator or an anti-unitary operator. The proof of this theorem is given in Appendix A.

A unitary operator U is such that

$$< U\psi|U\phi >=< \psi|U^\dagger U|\phi >=< \psi|\phi > \tag{4}$$

for all states $|\phi >, |\psi >$, so that

$$U^\dagger U = UU^\dagger = I \tag{5}$$

and is linear so that

$$U(\alpha|\phi > +\beta|\psi >) = \alpha U|\phi > +\beta U|\psi > \tag{6}$$

where α, β are arbitrary complex numbers. In that case, $\frac{|<U\psi|U\phi>|^2}{<U\psi|U\psi><U\phi|U\phi>} = \frac{|<\psi|U^\dagger U|\phi>|^2}{<\psi|U^\dagger U|\psi><\phi|U^\dagger U|\phi>} = \frac{|<\psi|\phi>|^2}{<\psi|\psi><\phi|\phi>}$ as required.

An anti-unitary operator is such that

$$< U\psi|U\phi >=< \psi|\phi >^*=< \phi|\psi > \tag{7}$$

and is anti-linear so that

$$U(\alpha|\phi > +\beta|\psi >) = \alpha^* U|\phi > +\beta^* U|\psi > \tag{8}$$

for arbitrary complex numbers α, β. In that case, $\frac{|<U\psi|U\phi>|^2}{<U\psi|U\psi><U\phi|U\phi>} = \frac{|<\phi|\psi>|^2}{<\psi|\psi><\phi|\phi>} = \frac{|<\psi|\phi>|^2}{<\psi|\psi><\phi|\phi>}$ as required.

In our description of a symmetry, we have thus far focused on two different descriptions of the same physical system. That is, we have considered two different observers related to each other by a symmetry transformation. This perspective is referred to as the passive view. A completely equivalent alternative is to focus on a single observer and envision acting on the physical system by the symmetry operation. This is referred to as the active view. The two points of view are complimentary to each other and either can be employed. For example, if the physics is invariant under rotations, one can either consider rotating the system by some angle about some direction or by using a new coordinate system (observer) rotated with respect to the original coordinates by the equal but opposite angle about the same direction. Either viewpoint corresponds to a symmetry operation and in both cases one gets the same results as in the original description of any physical measurement.

We now study the consequences of requiring that the physics be invariant under a particular set of coordinate transformations which correspond to translations in space and time and spatial rotations. Consider two inertial observers who label the same space-time point by $(\vec{r}, t)$ and $(\vec{r}\,', t')$. These observers will be referred to as $\mathcal{O}$ and $\mathcal{O}'$ respectively.

Space Translation Symmetry:

Spatial translation invariance dictates that the physics does not change if an experiment is performed at two different points in space or equivalently if the 2 observers are related by a spatial translation $\vec{a}$. Let me slightly elaborate. If you build any kind of apparatus to do any kind of experiment and then go ahead and build the same apparatus to do the same kind of experiment with similar things but put them here instead of there, i.e. merely translated from one place to another in space, then if the system is space translation invariant, the same thing will happen in the translated experiment as would happen in the original experiment. It is necessary in defining this idea to take into account moving everything that might have an influence on the experiment. Note that $\vec{a}$ is the same for all points. That is, the entire system is translated by the same uniform amount. Under the space translation, the coordinates of the 2 inertial observers are related by

$$\begin{aligned} x_i' &= x_i + a_i \\ t' &= t\,. \end{aligned} \tag{9}$$

There are 3 parameters, $\vec{a}$, which characterize the spatial translation. (One independent translation for each of the 3 spatial directions).

The translation transformation in space induces a transformation on the Hilbert space and there is an associated operator $U(a)$ which acts on the Hilbert space of states. Since the square of a translation is again a translation, we have that

$$U(a) = U^2(a/2) \tag{10}$$

Wigner's theorem asserts that each of the $U(a)$ is either a unitary or anti-unitary operator. But since square of either a unitary or anti-unitary operator is a unitary operator, we conclude that $U(a)$ must be a unitary operator. Note that this same type of argument can be employed to show that the quantum mechanical operator representing any continuous symmetry transformation is unitary.

Let us now consider the case when the parameters characterizing the translation transformation differ only infinitesimally from no translation so that

$$a_i = \epsilon_i \quad , \quad |\epsilon_i| << 1 \tag{11}$$

with the ϵ_i being 3 real parameters characterizing the infinitesimal space translations.

Since the transformation differs only infinitesimally from no transformation, it follows that the unitary operator $U(\epsilon)$ differs only infinitesimally from the identity operator and thus can be written as

$$U(\epsilon) = I - i\epsilon_i \frac{P_i}{\hbar} \tag{12}$$

Imposing the unitarity of U and retaining terms to first order in the small parameters gives

$$\begin{aligned} I &= U^\dagger(\epsilon)U(\epsilon) \\ &= [I + i\epsilon_i \frac{P_i^\dagger}{\hbar}][I - i\epsilon_i \frac{P_i}{\hbar}] \\ &= I - \frac{i}{\hbar}\epsilon_i(P_i - P_i^\dagger) \end{aligned} \tag{13}$$

Since ϵ_i are independent parameters, it follows that

$$P_i^\dagger = P_i \tag{14}$$

The 3 hermitian operators P_i are the called the generators of translations.

Now consider 2 successive such infinitesimal translations:

$$[U(\epsilon)]^2 = I - \frac{i}{\hbar} 2\epsilon_i P_i = U(2\epsilon) \tag{15}$$

Writing $\epsilon_i = \frac{a_i}{N}$ with the idea that $N \to \infty$, then

$$[U(\frac{a}{N})]^2 = U(\frac{2a}{N}) \tag{16}$$

Continue to compound the infinitesimal transformations so that

$$\lim_{N\to\infty}[U(\frac{a}{N})]^N = \lim_{N\to\infty} U(N\frac{a}{N}) = U(a) \tag{17}$$

Thus the unitary operator corresponding to the finite spatial translation is

$$\begin{aligned} U(a) &= \lim_{N\to\infty}[U(\frac{a}{N})]^N \\ &= \lim_{N\to\infty}[I - \frac{i}{\hbar}\frac{a_i}{N}P_i]^N \\ &= e^{-\frac{i}{\hbar}a_i P_i} \end{aligned} \tag{18}$$

Invariance under spatial translations dictates that

$$\psi'(\vec{r}) = \psi(\vec{r} - \vec{a}) \tag{19}$$

or

$$< \vec{r}|\psi' > = < \vec{r} - \vec{a}|\psi > \tag{20}$$

Using

$$|\psi' > = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}|\psi > \tag{21}$$

along with the Taylor expansion

$$< \vec{r} - \vec{a}| = e^{-\vec{a}\cdot\nabla} < \vec{r}| \tag{22}$$

gives

$$< \vec{r}|e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}|\psi > = e^{-\vec{a}\cdot\nabla} < \vec{r}|\psi > \tag{23}$$

or since $|\psi >$ is arbitrary that

$$< \vec{r}|e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}} = e^{-\vec{a}\cdot\nabla} < \vec{r}| \tag{24}$$

For infinitesimal $\vec{a}$ this reads

$$-\frac{i}{\hbar}\vec{a}\cdot < \vec{r}|\vec{P} = -\vec{a}\cdot\nabla < \vec{r}| \tag{25}$$

Consequently we recognize that the hermitian operator $\vec{P}$ generating space translations has the coordinate space representation

$$< \vec{r}|\vec{P} = \frac{\hbar}{i}\nabla < \vec{r}| \tag{26}$$

and can be identified as the momentum operator.

It follows from Eq. (26) that

$$[P_i, P_j] = 0 \tag{27}$$

An example

Let us see how the presence of a translation symmetry can facilitate solving for a Hamiltonian eigenfunction. Suppose we have a system which is invariant under translations along a particular direction, which we choose to be the z direction. Invariance under such translations then dictates that the Hamiltonian cannot depend on the z coordinate operator so the time independent Hamiltonian has the dependence $H = H(x, y, p_x, p_y, p_z)$. We shall employ the z translation symmetry to obtain the z dependence of a non-degenerate Hamiltonian eigenfunction.
Since p_z commutes with x, y, p_x, p_y, p_z and H is independent of z it follows that $[H, p_z] = 0$.

Let E be a non-degenerate H eigenvalue with eigenfunction $\psi(\vec{r})$:

$$H\psi(\vec{r}) = E\psi(\vec{r}) \tag{28}$$

It follows that

$$0 = [H, p_z]\psi(\vec{r}) \tag{29}$$

so that

$$H\Big(p_z\psi(\vec{r})\Big) = p_z H\psi(\vec{r}) = E\Big(p_z\psi(\vec{r})\Big) \tag{30}$$

Thus $p_z\psi(\vec{r}) = \frac{\hbar}{i}\frac{\partial\psi(\vec{r})}{\partial z}$ is an H eigenstate with eigenvalue E.

Since the eigenvalue E is non-degenerate, it follows that the eigenfunction $\frac{\hbar}{i}\frac{\partial\psi(\vec{r})}{\partial z}$ corresponds to the same state as the eigenfunction $\psi(\vec{r})$. Thus these two functions are proportional to each other:

$$\frac{\hbar}{i}\frac{\partial\psi(\vec{r})}{\partial z} = \hbar k_z\psi(\vec{r}) \tag{31}$$

Solving this p_z eigenvalue equation yields

$$\psi(\vec{r}) = e^{ik_z z} f(x,y) \tag{32}$$

with $-\infty < k_z < \infty$.

Time Translation Symmetry:

If the physics does not change if the experiment is performed at a different time so that the 2 observers are related by a time translation b, then the system is said to be time translation invariant. Note that it is the entire system which is translated by the same uniform temporal amount. Under the time translation, the coordinates of the 2 inertial observers are related by

$$\begin{aligned} x_i' &= x_i \\ t' &= t + b \end{aligned} \tag{33}$$

so that there is 1 parameter, b, characterizing the time translation.

The time translation transformation induces a transformation on the Hilbert space and there is an associated operator $U(b)$ which acts on the Hilbert space of states. Since any time translation can be achieved by compounding infinitesimal time transformations, we know that $U(b)$ is a unitary operator.

Let us now consider the case when the parameter characterizing the time translation transformation differ only infinitesimally from no translation so that

$$b = \beta \quad , \quad |\beta| << 1 \tag{34}$$

with β is 1 real parameter characterizing the infinitesimal time translations.

Since the transformation differs only infinitesimally from no transformation, it follows that the unitary operator $U(\beta)$ differs only infinitesimally from the identity operator and thus can be written as

$$U(\beta) = I + i\beta \frac{H}{\hbar} \tag{35}$$

Imposing the unitarity of U and retaining terms to first order in the small parameters gives

$$I = U^\dagger(\beta) U(\beta)$$

$$\begin{aligned} &= [I - i\beta\frac{H^\dagger}{\hbar}][I + i\beta\frac{H}{\hbar}] \\ &= I + \frac{i}{\hbar}\beta(H - H^\dagger) \end{aligned} \tag{36}$$

Since β is an independent parameter, it follows that

$$H^\dagger = H \tag{37}$$

The hermitian operators H is the called the generators of translations.

For infinitesimal time translations, $b = \beta$,

$$U(\beta) = I + \frac{i}{\hbar}\beta H \tag{38}$$

Consider 2 such infinitesimal translations:

$$[U(\beta)]^2 = I + \frac{i}{\hbar}2\beta H = U(2\beta) \tag{39}$$

Writing $\beta = \frac{b}{N}$ with the idea that $N \to \infty$, then

$$[U(\frac{b}{N})]^2 = U(\frac{2b}{N}) \tag{40}$$

Continue to compound the infinitesimal transformations so that

$$\lim_{N\to\infty}[U(\frac{b}{N})]^N = \lim_{N\to\infty} U(N\frac{b}{N}) = U(b) \tag{41}$$

Thus the unitary operator corresponding to the finite time translation is

$$\begin{aligned} U(b) &= \lim_{N\to\infty}[U(\frac{b}{N})]^N \\ &= \lim_{N\to\infty}[I + \frac{i}{\hbar}\frac{b}{N}H]^N \\ &= e^{\frac{i}{\hbar}bH} \end{aligned} \tag{42}$$

Now consider an observer $\mathcal{O}$ and the time translated by b observer $\mathcal{O}'$. They describe the state of the system by $|\psi>$ and $|\psi'>$ respectively, where

$$|\psi'> = e^{\frac{i}{\hbar}Hb}|\psi> \tag{43}$$

with $H = H^\dagger$ the generator of time translations. Here we have been employing the passive description in which two observers related by a symmetry transformation are describing the same physics. Alternatively, we could employ the active view in which there is a single observer, but the system itself undergoes the symmetry transformation. Using this viewpoint for the case of time translations, then $|\psi' >\equiv |\psi(t)>$ is the state which has evolved from the state $|\psi >= |\psi(0)>$ after the time interval $t = -b$ has passed and the states are related as

$$|\psi(t)>= e^{-\frac{i}{\hbar}Ht}|\psi(0)> \tag{44}$$

It follows that

$$i\hbar\frac{d}{dt}|\psi(t)>= H|\psi(t)> \tag{45}$$

and we can identify the generator of time translations, H, as the Hamiltonian of the system.

Schrödinger and Heisenberg Representations

Thus far in describing a closed physical system, we have employed a formalism in which all operators are time independent while the physical states have time dependence governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)>= H|\psi(t)> \ . \tag{46}$$

This dynamical description is referred to as the Schrödinger representation or picture. Such a description, however, is not unique. One could alternatively describe the closed system using time independent states and time dependent operators. Such a description is referred to as the Heisenberg representation or picture. It is related to the Schrödinger representation by performing a unitary transformation.

Let $|\psi(t)>_S$ and A_S denote a physical state vector (i.e solution to the time dependent Schrödinger equation) and operator in the Schrödinger representation. Now define the corresponding state and operator in the Heisenberg representation by

$$|\psi>_H= e^{\frac{i}{\hbar}Ht}|\psi(t)>_S \tag{47}$$

$$A_H(t) = e^{\frac{i}{\hbar}Ht}A_S e^{-\frac{i}{\hbar}Ht} \tag{48}$$

where $H = H^\dagger$ is the hermitian Hamiltonian operator. Note that $H = H_S = H_H$. The operator

$$U(t) = e^{\frac{i}{\hbar}Ht} \tag{49}$$

is clearly unitary since

$$U^{-1}(t) = U^\dagger(t) = e^{-\frac{i}{\hbar}Ht}. \tag{50}$$

The scalar product of two vectors in the Heisenberg representation

$$_H < \psi|\phi >_H = \ _S < \psi(t)|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht}|\phi(t) >_S = \ _S < \psi(t)|\phi(t) >_S \tag{51}$$

is equal to the scalar product in the Schrödinger representation. Moreover since

$$\begin{aligned} _H < \psi|A_H(t)|\phi >_H &= \ _S < \psi(t)|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht}A_S e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht}|\phi(t) >_S \\ &= \ _S < \psi(t)|A_S|\phi(t) >_S , \end{aligned} \tag{52}$$

we see that matrix elements of the corresponding operators in the two pictures are also identical. The operators $A_H(t)$ and A_S are unitarily equivalent. Now consider

$$\begin{aligned} \frac{d}{dt}A_H(t) &= \frac{d}{dt}(e^{\frac{i}{\hbar}Ht}A_S e^{-\frac{i}{\hbar}Ht}) \\ &= \frac{i}{\hbar}e^{\frac{i}{\hbar}Ht}(HA_S - A_S H)e^{-\frac{i}{\hbar}Ht} \\ &= \frac{i}{\hbar}(HA_H(t) - A_H(t)H) \\ &= \frac{i}{\hbar}[H, A_H(t)] \end{aligned} \tag{53}$$

which is the Heisenberg equation of motion.
On the other hand,

$$\begin{aligned} \frac{d}{dt}|\psi >_H &= \frac{d}{dt}[e^{\frac{i}{\hbar}Ht}|\psi(t) >_S] \\ &= \frac{i}{\hbar}e^{\frac{i}{\hbar}Ht}H|\psi(t) >_S + e^{\frac{i}{\hbar}Ht}\frac{d}{dt}|\psi(t) >_S \\ &= \frac{i}{\hbar}e^{\frac{i}{\hbar}Ht}[H|\psi(t) >_S - i\hbar\frac{d}{dt}|\psi(t) >_S] \\ &= 0 \end{aligned} \tag{54}$$

as a consequence of the Schrödinger equation. Thus the state vectors in the Heisenberg representation are time independent while the operators have time dependence given by the Heisenberg equation of motion. Further note that $A_H(0) = A_S$ and $|\psi(0) >_S = |\psi >_H$ so that the two representations coincide at $t = 0$.

Note that since it is unitary operator which connects the Schrödinger and Heisenberg pictures, the Heisenberg operators also satisfy canonical commutation relations at each time t:

$$\begin{aligned} &[x_{iH}(t), p_{jH}(t)] = i\hbar\delta_{ij} \\ &[x_{iH}(t), x_{jH}(t)] = 0 = [p_{iH}(t), p_{jH}(t)] \end{aligned} \tag{55}$$

Further note that for the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}) \tag{56}$$

the Heisenberg equations of motion for the coordinate and momentum operators reduce to

$$\begin{aligned} \frac{dx_{iH}(t)}{dt} &= \frac{1}{m} p_{iH}(t) \\ \frac{dp_{iH}(t)}{dt} &= -\frac{\partial V(\vec{r}_H(t))}{\partial x_{iH}(t)} \end{aligned} \tag{57}$$

which have the same structure as the classical equations of motion.

Recall the position operator eigenvalue problem which in the Schrödinger representation takes the form

$$\vec{r}_S|\vec{r}\,' >_S = \vec{r}\,'|\vec{r}\,' >_S \tag{58}$$

Here $|\vec{r}\,' >_S$ is time independent ket-vector. Now define the position operator in Heisenberg representation as

$$\vec{r}_H(t) = e^{\frac{i}{\hbar}Ht}\vec{r}_S e^{-\frac{i}{\hbar}Ht} \tag{59}$$

which has the eigenvalue equation (c.f. Eq. (58))

$$\vec{r}_H(t)(e^{\frac{i}{\hbar}Ht}|\vec{r}\,' >_S) = \vec{r}\,'(e^{\frac{i}{\hbar}Ht}|\vec{r}\,' >_S) \tag{60}$$

Thus we can define the Heisenberg representation coordinate state eigenvector at time t as

$$|\vec{r}\,';t>_H = e^{\frac{i}{\hbar}Ht}|\vec{r}\,'>_S \tag{61}$$

which satisfies

$$\vec{r}_H(t)|\vec{r}\,';t>_H = \vec{r}\,'|\vec{r}\,';t>_H \tag{62}$$

Note that the label t appearing in the $|\vec{r}\,';t>_H$ is there to indicate that the vector $|\vec{r}\,';t>_H$ is an eigenstate of $\vec{r}_H(t)$ at time t with eigenvalue $\vec{r}\,'$. On the other hand, it is not an eigenstate of $\vec{r}_H(t_0)$ if $t_0 \neq t$.

We also have that

$$\begin{aligned}
<\vec{r};t|\vec{p}(t) &= <\vec{r}|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Ht}pe^{-\frac{i}{\hbar}Ht} \\
&= <\vec{r}|pe^{-\frac{i}{\hbar}Ht} \\
&= \frac{\hbar}{i}\nabla <\vec{r}|e^{-\frac{i}{\hbar}Ht} \\
&= \frac{\hbar}{i}\nabla <\vec{r};t|
\end{aligned} \tag{63}$$

As an example, consider the 1-dimensional simple harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 = \hbar\omega_0(a^\dagger a + \frac{1}{2}) \tag{64}$$

Using the Schrödinger picture, the time evolution of the state vector is obtained as the solution to the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)>_S = H|\psi(t)>_S \tag{65}$$

as

$$|\psi(t)>_S = e^{-\frac{i}{\hbar}Ht}|\psi(0)>_S = \sum_{n=0}^{\infty} c_n e^{-\frac{i}{\hbar}E_nt}|n> \tag{66}$$

where in obtaining the second equality, we have expanded the state vector $|\psi(0)>_S$ in terms of the complete set of H eigenstates as $|\psi(0)>_S = \sum_{n=0}^{\infty} c_n|n>$ where $|n>$ is an H eigenstate satisfying $H|n> = E_n|n>$.

On the other hand, in the Heisenberg picture, we have

$$|\psi>_H = e^{\frac{i}{\hbar}Ht}|\psi(t)>_S = |\psi(0)>_S \tag{67}$$

The annihilation operator in the Heisenberg picture is

$$a_H(t) = e^{\frac{i}{\hbar}Ht} a_S e^{-\frac{i}{\hbar}Ht} = e^{\frac{i}{\hbar}Ht} a e^{-\frac{i}{\hbar}Ht} \quad ; \quad a_S \equiv a \tag{68}$$

so that $a_H(0) = a_S = a$. It follows that

$$\begin{aligned} \frac{da_H(t)}{dt} &= \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht}[H,a]e^{-\frac{i}{\hbar}Ht} \\ &= -i\omega_0 e^{\frac{i}{\hbar}Ht} a e^{-\frac{i}{\hbar}Ht} \\ &= -i\omega_0 a_H(t) \end{aligned} \tag{69}$$

Integrating this equation gives

$$a_H(t) = e^{-i\omega_0 t} a_H(0) = e^{-i\omega_0 t} a \tag{70}$$

This result can also be secured by integrating the Heisenberg equation of motion

$$\begin{aligned} \frac{da_H(t)}{dt} &= \frac{i}{\hbar}[H, a_H(t)] \\ &= \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht}[H,a]e^{-\frac{i}{\hbar}Ht} \\ &= -i\omega_0 a_H(t) \end{aligned} \tag{71}$$

yielding

$$a_H(t) = e^{-i\omega_0 t} a \tag{72}$$

Taking the hermitian conjugate gives

$$a_H^\dagger(t) = e^{i\omega_0 t} a^\dagger \tag{73}$$

It then follows that

$$\begin{aligned} x_H(t) &= \sqrt{\frac{\hbar}{2m\omega_0}}(a_H(t) + a_H^\dagger(t)) \\ &= x\ cos(\omega_0 t) + \frac{1}{m\omega_0} p\ sin(\omega_0 t) \end{aligned} \tag{74}$$

and

$$p_H(t) = \frac{1}{i}\sqrt{\frac{\hbar\omega_0 m}{2}}(a_H(t) - a_H^\dagger(t))$$

$$= \quad p\ cos(\omega_0 t) - m\omega_o x\ sin(\omega_0 t) \tag{75}$$

In obtaining this result, we used that

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger) \\ p &= \frac{1}{i}\sqrt{\frac{\hbar m\omega_0}{2}}(a - a^\dagger) \end{aligned} \tag{76}$$

Space Rotations

Isotropy of space dictates that the physics does not change if the experiment is performed by two different observers who are related by a spatial rotation through an angle θ in the direction $\hat{\theta}$. Note that $\vec{\theta} = \hat{\theta}\theta$ is the same for all points. That is, the entire system is rotated by the same uniform amount. Consider two observers who are related by a rotation and have a common origin so that we can write the relation between the two inertial observers as

$$x'_i = R_{ij}(\vec{\theta})x_j \tag{77}$$

with R_{ij} real. A rotation has the property that it leaves the distance between 2 points unchanged. Thus

$$\vec{r}\,'^{\,2} = \vec{r}^2 \Rightarrow x'_i x'_i = x_i x_i \tag{78}$$

or

$$R_{ij}x_j R_{ik}x_k = x_j x_j = \delta_{jk}x_j x_k \tag{79}$$

and since this must hold for all x_i, that

$$R_{ij}R_{ik} = \delta_{jk} \tag{80}$$

Introducing the 3×3 matrix R whose ij^{th} component is R_{ij}, this matrix satisfies

$$RR^T = R^T R = I \tag{81}$$

where the superscript T denotes transposition. Thus the matrix R is a real orthogonal matrix and

$$R^{-1} = R^T \tag{82}$$

or in components

$$(R^{-1})_{ij} = R_{ji} \tag{83}$$

It follows that matrix R has 3 independent parameters which are used to characterize the rotation. These can be taken to be the 3 rotation angles $\vec{\theta}$ or the 3 Euler angles. We shall use the angles θ_i so that $R = R(\vec{\theta})$. The explicit dependence of $R(\vec{\theta})$ on $\vec{\theta}$ can be obtained using the simple geometry displayed in Appendix B.

Let us now consider the case when the parameters characterizing the rotation differ only infinitesimally from no rotation so that

$$R_{ij} = \delta_{ij} + \omega_{ij} \quad , \quad |\omega_{ij}| << 1 \tag{84}$$

with ω_{ij} real parameters. The orthogonality of R, $R^T = R^{-1}$, restricts the infinitesimal parameters ω_{ij}. Working to first order in the ω,

$$(R^{-1})_{ij} = \delta_{ij} - \omega_{ij} \tag{85}$$

while

$$(R^T)_{ij} = \delta_{ij} + \omega_{ji}\,. \tag{86}$$

Thus

$$R^T = R^{-1} \Rightarrow \omega_{ij} = -\omega_{ji} \quad \text{(antisymmetric)} \tag{87}$$

and there are only 3 independent parameters characterizing the infinitesimal rotations.

The rotation in space induces transformations on the Hilbert space and there is an associated operator $U(1+\omega)$ which acts on the Hilbert space of states. Since any rotation can be achieved by compounding infinitesimal rotations, we know that $U(1+\omega)$ is a unitary operator. For rotations differing only infinitesimally from no rotation, it follows that the unitary operator $U(1+\omega)$ differs only infinitesimally from the identity operator and thus can be written as

$$U(1+\omega) = I + \frac{i}{2}\omega_{ij}\frac{J_{ij}}{\hbar} \tag{88}$$

Since $\omega_{ij} = -\omega_{ji}$, we can take $J_{ij} = -J_{ji}$ with no loss of generality. Imposing the unitarity of U and retaining terms to first order in the small parameters gives

$$\begin{aligned} I &= U^\dagger(1+\omega)U(1+\omega) \\ &= [I - \frac{i}{2}\omega_{ij}\frac{J_{ij}^\dagger}{\hbar}][I + \frac{i}{2}\omega_{ij}\frac{J_{ij}}{\hbar}] \\ &= I + \frac{i}{2\hbar}\omega_{ij}(J_{ij} - J_{ij}^\dagger) \end{aligned} \tag{89}$$

Since $\omega_{ij} = -\omega_{ji}$ are independent parameters, it follows that

$$J_{ij}^\dagger = J_{ij} \tag{90}$$

First consider a rotation by an angle θ about the $\hat{z}$ axis so that

$$R(\theta\hat{z}) = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{91}$$

For infinitesimal θ, this takes the form $R_{ij} = \delta_{ij} + \omega_{ij}(\theta\hat{z})$ with

$$\omega(\theta\hat{z}) = \theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{92}$$

ie. $\omega_{12} = -\theta, \quad \omega_{21} = \theta$.

Note that since $\omega_{ij} = -\omega_{ji}$, we can always write $\omega_{ij} = -\epsilon_{ijk}\theta_k$. For infinitesimal rotations about the z axis, $\theta_k = \theta\delta_{k3}$ so that only $\omega_{12} = -\omega_{21} = -\theta$ are non-zero.

Thus for an infinitesimal rotation by θ about the $\hat{z}$-axis,

$$\begin{aligned} U(1+\omega(\theta\hat{z})) &= I + \frac{i}{2\hbar}\omega_{ij}J_{ij} \\ &= I + \frac{i}{\hbar}\omega_{12}J_{12} \\ &= I - \frac{i}{\hbar}\theta J_{12} \end{aligned} \tag{93}$$

Defining

$$\vec{J} = (J_{23}, J_{31}, J_{12}) \tag{94}$$

that is,

$$J_i = \frac{1}{2}\epsilon_{ijk} J_{jk} \;\; \Leftrightarrow J_{ij} = \epsilon_{ijk} J_k \tag{95}$$

it follows that

$$U(1 + \omega(\theta\hat{z})) = I - \frac{i}{\hbar}\theta J_z \tag{96}$$

Employing the same reasoning as in the case of translations, we can compound the infinitesimal rotations about the $\hat{z}$-axis to construct a finite rotation about this axis of angle θ which is represented on the Hilbert space by the unitary operator

$$U(R(\theta\hat{z})) = e^{-\frac{i}{\hbar}\theta J_z} \tag{97}$$

Calling the rotation axis $\hat{z}$ was an arbitrary choice and we could just as well have labeled the direction $\hat{\theta}$. It would then follow that the rotation of angle θ about the direction $\hat{\theta}$ is represented by the unitary operator

$$U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\theta}} \tag{98}$$

The hermitian operators J_i generate spatial rotations. Thus under a rotation by angle θ in the direction $\hat{\theta}$, the rotated observer describes the state $|\psi>$ as the state

$$|\psi'>= U(R(\vec{\theta}))|\psi>= e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\theta}}|\psi> \tag{99}$$

Since $<\psi|J_i|\psi>$ transforms as a vector under rotations (i.e. it transforms in the same way as the spatial coordinates) it follows that

$$<\psi'|J_j|\psi'>= R_{jk}(\vec{\theta}) <\psi|J_k|\psi> \tag{100}$$

Using that

$$|\psi'>= U(R(\vec{\theta}))|\psi> \tag{101}$$

it follows that

$$<\psi|U^{-1}(R(\vec{\theta}))J_j U(R(\vec{\theta}))|\psi>= R_{jk}(\vec{\theta}) <\psi|J_k|\psi> \tag{102}$$

and since the state $|\psi>$ is arbitrary, that

$$U^{-1}(R(\vec{\theta}))J_j U(R(\vec{\theta})) = R_{jk}(\vec{\theta}) J_k \tag{103}$$

or

$$e^{\frac{i}{\hbar}\vec{J}\cdot\vec{\theta}}J_j e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\theta}} = R_{jk}(\vec{\theta})J_k \tag{104}$$

Now consider an infinitesimal rotation so that $\theta \to \delta\theta$ with $|\delta\theta| << 1$.
We can then Taylor expand $e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\delta\theta}} = 1 - \frac{i}{\hbar}J_i\delta\theta_i$ and $R_{jk}(\vec{\delta\theta}) = \delta_{jk} + \omega_{jk} = \delta_{jk} - \epsilon_{jki}\delta\theta_i$. Substituting into the above then gives (retaining terms through linear in $\delta\theta$)

$$\frac{i}{\hbar}\delta\theta_i[J_i, J_j] = -\delta\theta_i\epsilon_{jki}J_k \tag{105}$$

or, since the $\delta\theta_i$ are independent, that

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{106}$$

which is the angular momentum algebra.

In an analogous fashion, we have that

$$< \psi'|P_j|\psi' >= R_{jk}(\vec{\theta}) < \psi|P_k|\psi > \tag{107}$$

leading to the commutation relations

$$[J_i, P_j] = i\hbar\epsilon_{ijk}P_k \tag{108}$$

Note that one realization of these commutation relations is obtained using the differential operators

$$\begin{aligned} P_i &= \frac{\hbar}{i}\frac{\partial}{\partial x_i} \\ J_i &= \epsilon_{ijk}x_j\frac{\hbar}{i}\frac{\partial}{\partial x_k} = L_i \text{ (orbital angular momentum)} \end{aligned} \tag{109}$$

We thus secure the algebra for a space and time translation and rotationally invariant system

$$\begin{aligned} [J_i, H] &= 0 \\ [P_i, H] &= 0 \\ [P_i, P_j] &= 0 \\ [J_i, J_j] &= i\epsilon_{ijk}J_k \\ [J_i, P_j] &= i\epsilon_{ijk}\hbar P_k \end{aligned} \tag{110}$$

One Hamiltonian satisfying this algebra is a free particle: $H = \frac{\vec{p}^2}{2m}$; another is the 2-body system: $H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$

Recall that in the Heisenberg picture, an operator $A(t)$ has a time dependence (here we assume that the operator A is time independent in the Schrödinger picture) given by the Heisenberg equation of motion

$$-i\hbar\frac{dA(t)}{dt} = [H, A(t)] \tag{111}$$

It follows that the constants of motion are those operators which commute with H. Thus for a system invariant under space and time translation and rotations, the momentum, $\vec{P}$, and the angular momentum, $\vec{J}$, are constants of the motion.

Time translation invariance $\Leftrightarrow$ H constant of motion $\Leftrightarrow$ energy conserved

Space translation invariance $\Leftrightarrow$ $\vec{P}$ constant of motion $\Leftrightarrow$ momentum conserved

Space rotation invariance $\Leftrightarrow$ $\vec{J}$ constant of motion $\Leftrightarrow$ angular momentum conserved.

Since the constants of the motion commute with H, they can be simultaneously diagonalized along with H. For those operators which also mutually commute, they can be used as (part of) the CSCO. As such their eigenvalues can be employed in the labeling for a basis of the state space.

Appendix A: Proof of Wigner's Theorem

The fundamental requirement on the operator U is that

$$\frac{|<U\psi|U\phi>|^2}{<U\psi|U\psi><U\phi|U\phi>} = \frac{|<\psi|\phi>|^2}{<\psi|\psi><\phi|\phi>} \tag{112}$$

First, by rescaling $|U\psi> \rightarrow \sqrt{\frac{<U\psi|U\psi>}{<\psi|\psi>}}|U\psi>$, this fundamental requirement takes the form

$$|<U\psi|U\phi>| = |<\psi|\phi>| \tag{113}$$

Now introduce a complete orthonormal basis $\{|\phi_k>\}$ satisfying

$$<\phi_k|\phi_l>= \delta_{kl} \tag{114}$$

The fundamental requirement gives

$$|<U\phi_k|U\phi_l>| = |<\phi_k|\phi_l>| = 0 \quad , \quad k \neq l \tag{115}$$

and

$$<U\phi_k|U\phi_k>=<\phi_k|\phi_k>= 1 \tag{116}$$

Taken together these reduce to

$$<U\phi_k|U\phi_l>= \delta_{kl} \tag{117}$$

so the $\{|U\phi_k>\}$ also form a complete orthonormal set of states. An arbitrary ket $|\psi>$ can be expanded in terms of the $\{|\phi_k>\}$ as

$$|\psi>= \sum_k \psi_k|\phi_k> \tag{118}$$

with $\psi_k =<\phi_k|\psi>$, while the state $U|\psi>= |U\psi>$ can be expanded in terms of the $\{|U\phi_k>\}$ as

$$U|\psi>= |U\psi>= \sum_k \psi'_k|U\phi_k> \tag{119}$$

with $\psi'_k =<U\phi_k|U\psi>$. The fundamental requirement then dictates that

$$|\psi'_k| = |\psi_k| \tag{120}$$

Let $|\phi_1>$ be a basis state which has a non-trivial coefficient in the expansion of $|\psi>$ and consider the states $|\phi_1> + |\phi_k>$, with $k = 2, 3, ...$ Expanding the states $|U(\phi_1 + \phi_k)>$ in terms of the $\{|U\phi_l>\}$ gives

$$|U(\phi_1 + \phi_k) >= \sum_l \alpha_{lk} |U\phi_l> \tag{121}$$

with $\alpha_{lk} =< U\phi_l|U(\phi_1 + \phi_k)>$, $k = 2, 3, ...$ Using the fundamental requirement that

$$\begin{aligned} |< U\phi_l|U(\phi_1 + \phi_k)>| &= |< \phi_l|(\phi_1 + \phi_k)>| \\ &= |< \phi_l|\phi_1> + < \phi_l|\phi_k>| \\ &= |\delta_{l1} + \delta_{lk}| \end{aligned} \tag{122}$$

it follows that $|\alpha_{lk}| = |\delta_{l1} + \delta_{lk}|$, $k = 2, 3, ..,$ so that only the l values of 1 and k can appear in the expansion of $U(|\phi_1> + |\phi_k>)$ in terms of the $\{|U\phi_l>\}$ giving

$$|U(\phi_1 + \phi_k) >= \alpha_k |U\phi_1> + \beta_k |U\phi_k> \tag{123}$$

where we have defined $\alpha_k \equiv \alpha_{1k}$ and $\beta_k \equiv \alpha_{kk}$. Since $|\alpha_k| = |\beta_k| = 1$, it follows that α_k , β_k are pure phases.

Now introduce U' by defining

$$\begin{aligned} U'|\phi_1> &= |U'\phi_1>=|U\phi_1> \\ U'|\phi_k> &= |U'\phi_k>=\frac{\beta_k}{\alpha_k}|U\phi_k> \\ |U'(\phi_1 + \phi_k)> &= \frac{1}{\alpha_k}|U(\phi_1 + \phi_k)> \end{aligned} \tag{124}$$

which specifies the phases. So doing Eq. (123) takes the form

$$|U'(\phi_1 + \phi_k) >= |U'\phi_1> + |U'\phi_k> \tag{125}$$

Now recall U' by U so that

$$|U(\phi_1 + \phi_k) >= |U\phi_1> + |U\phi_k> \tag{126}$$

Now consider

$$\begin{aligned}
| < U(\phi_1 + \phi_k)|U\psi > | &= |\sum_l \psi'_l < U(\phi_1 + \phi_k)|U\phi_l > | \\
&= |\sum_l \psi'_l(< U\phi_1| + < U\phi_k)|U\phi_l > | \\
&= |\sum_l \psi'_l(\delta_{1l} + \delta_{kl})| \\
&= |\psi'_1 + \psi'_k| \quad , \quad k = 2, 3, ... \qquad (127)
\end{aligned}$$

and

$$\begin{aligned}
| < \phi_1 + \phi_k|\psi > | &= |\sum_l \psi_l < \phi_1 + \phi_k|\phi_l > | \\
&= |\sum_l \psi_l(\delta_{1l} + \delta_{kl})| \\
&= |\psi_1 + \psi_k| \quad , \quad k = 2, 3, ... \qquad (128)
\end{aligned}$$

As a consequence of the fundamental requirement, $| < U(\phi_1 + \phi_k)|U\psi > | = | < \phi_1 + \phi_k|\psi > |$, we thus secure

$$|\psi'_1 + \psi'_k| = |\psi_1 + \psi_k| \quad , \quad k = 2, 3, ... \qquad (129)$$

Recall that we already know that

$$|\psi'_l| = |\psi_l| \qquad (130)$$

Thus writing

$$\begin{aligned}
\psi_l &= |\psi_l|e^{iarg(\psi_l)} \\
\psi'_l &= |\psi'_l|e^{iarg(\psi'_l)} \qquad (131)
\end{aligned}$$

it follows that

$$\begin{aligned}
|\psi'_1 + \psi'_k| &= ||\psi_1|e^{iarg(\psi'_1)} + |\psi_k|e^{iarg(\psi'_k)}| \\
&= ||\psi'_1| + |\psi'_k|e^{i(arg(\psi'_k) - arg(\psi'_1))}| \\
&= \sqrt{(|\psi'_1| + |\psi'_k|cos^2(arg(\psi'_k) - arg(\psi'_1)) + |\psi'_k|^2 sin^2(arg(\psi'_k) - arg(\psi'_1))} \\
&= \sqrt{|\psi'_1|^2 + |\psi'_k|^2 + 2|\psi'_1||\psi'_k|cos(arg(\psi'_k) - arg(\psi'_1))} \\
&= \sqrt{|\psi_1|^2 + |\psi_k|^2 + 2|\psi_1||\psi_k|cos(arg(\psi'_k) - arg(\psi'_1))} \qquad (132)
\end{aligned}$$

Similarly

$$|\psi_1 + \psi_k| = \sqrt{|\psi_1|^2 + |\psi_k|^2 + 2|\psi_1||\psi_k|cos(arg(\psi_k) - arg(\psi_1))} \tag{133}$$

Application of Eq. (129) then gives

$$cos(arg(\psi'_k) - arg(\psi'_1)) = cos(arg(\psi_k) - arg(\psi_1)) \tag{134}$$

or

$$arg(\psi'_k) - arg(\psi'_1) = \pm(arg(\psi_k) - arg(\psi_1)) \tag{135}$$

Now consider

$$\begin{aligned} \frac{\psi'_k}{\psi'_1} &= \frac{|\psi'_k|e^{iarg(\psi'_k)}}{|\psi'_1|e^{iarg(\psi'_1)}} \\ &= \frac{|\psi'_k|}{|\psi'_1|}e^{i(arg(\psi'_k)-arg(\psi'_1))} \\ &= \frac{|\psi_k|}{|\psi_1|}e^{\pm i(arg(\psi_k)-arg(\psi_1))} \end{aligned} \tag{136}$$

or

$$\frac{\psi'_k}{\psi'_1} = \left\{ \begin{array}{c} \frac{\psi_k}{\psi_1} \\ (\frac{\psi_k}{\psi_1})^* \end{array} \right\} \tag{137}$$

Focus on the two possibilities in turn. First of all, we are free to choose the overall phase of $|\psi >$ and $|U\psi >$ so that $\psi'_1 = \psi_1$ is real. So doing, we find for the two alternatives:

(i) $\psi'_l = \psi_l$

In this case

$$U|\psi >= |U\psi >= \sum_l \psi'_l|U\phi_l >= \sum_l \psi_l U|\phi_l > \tag{138}$$

so that

$$U\sum_l \psi_l|\phi_l >= \sum_l \psi_l U|\phi_l > \tag{139}$$

(ii) $\psi'_l = \psi^*_l$

It then follows that

$$U|\psi>=|U\psi>=\sum_l \psi_l'|U\phi_l>=\sum_l \psi_l^* U|\phi_l> \tag{140}$$

and thus

$$U\sum_l \psi_l|\phi_l>=\sum_l \psi_l^* U|\phi_l> \tag{141}$$

Consider the two alternatives in more detail:

(i)

$$\begin{aligned} <U\chi|U\psi> &= <U\sum_k \chi_k\phi_k|U\sum_l \psi_l\phi_l> \\ &= <\sum_k \chi_k U\phi_k|\sum_l \psi_l U\phi_l> \\ &= \sum_{k,l} \chi_k^*\psi_l <U\phi_k|U\phi_l> \\ &= \sum_k \chi_k^*\psi_k \\ &= <\chi|\psi> \end{aligned} \tag{142}$$

while

$$\begin{aligned} U(\alpha|\chi>+\beta|\psi>) &= U(\sum_k(\alpha\chi_k+\beta\psi_k)|\phi_k>) \\ &= \alpha U\sum_k \chi_k|\phi_k>+\beta U\sum_k \psi_k|\phi_k> \\ &= \alpha U|\chi>+\beta U|\psi> \end{aligned} \tag{143}$$

so that U is a linear operator. Combining the two conditions, it follows that alternative (i) leads to U being a unitary operator.

(ii)

$$\begin{aligned} <U\chi|U\psi> &= <U\sum_k \chi_k\phi_k|U\sum_l \psi_l\phi_l> \\ &= <\sum_k \chi_k^* U\phi_k|\sum_l \psi_l^* U\phi_l> \end{aligned}$$

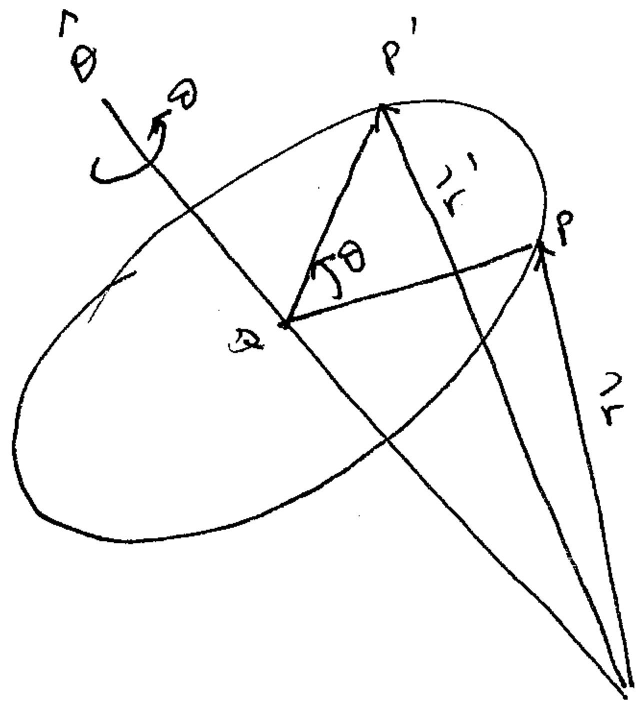
$$
\begin{aligned}
&= \sum_{k,l} \chi_k \psi_l^* < U\phi_k | U\phi_l > \\
&= \sum_k \chi_k \psi_k^* \\
&= < \psi | \chi > \\
&= < \chi | \psi >^* \qquad (144)
\end{aligned}
$$

while

$$
\begin{aligned}
U(\alpha|\chi > + \beta|\psi >) &= U(\sum_k (\alpha\chi_k + \beta\psi_k)|\phi_k >) \\
&= \alpha^* U \sum_k \chi_k |\phi_k > + \beta^* U \sum_k \psi_k |\phi_k > \\
&= \alpha^* U|\chi > + \beta^* U|\psi > \qquad (145)
\end{aligned}
$$

so that U is an anti-linear operator. Combining the two conditions, it follows that alternative (ii) leads to U being an anti-unitary operator.

Appendix B: Rotation matrix



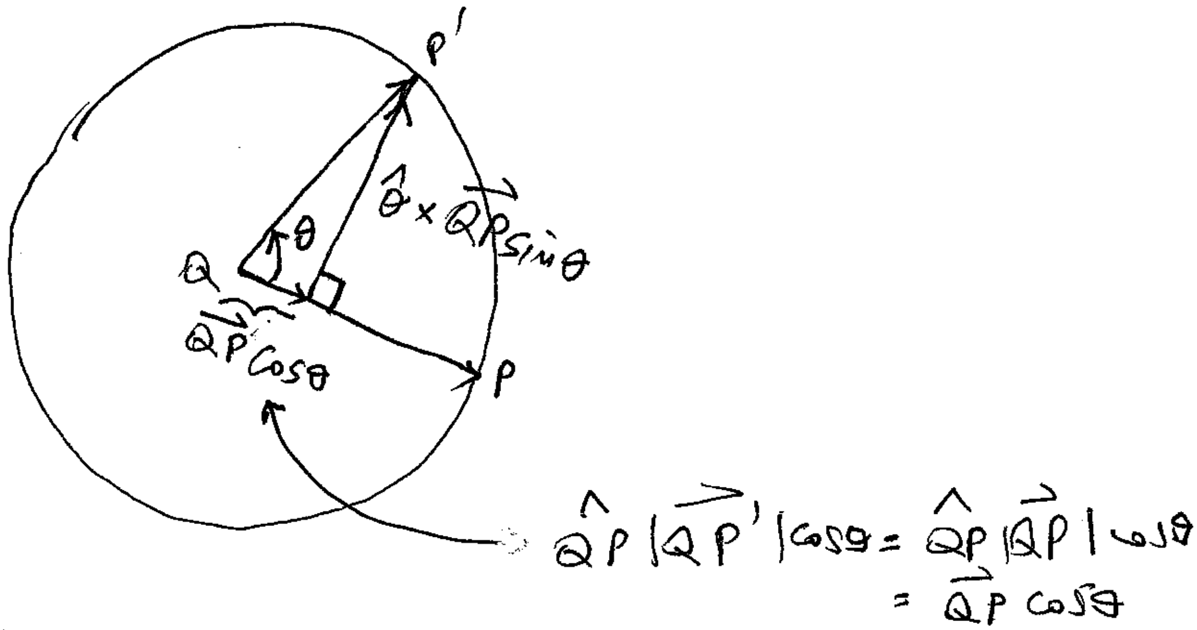
From the figure, it follows that

$$\vec{r}\,' = \overline{OQ} + \overline{QP'} \tag{146}$$

Since $\vec{r} = \overline{OP}$ and $\vec{r}\,' = \overline{OP'}$ are related by a rotation, we know that

$$|\overline{OP}| = |\overline{OP'}| \tag{147}$$

Looking down the $\hat{\theta}$ axis, we have using $|\overline{QP}| = |\overline{QP'}|$



that

$$\overline{QP'} = \overline{QP} cos\theta + \hat{\theta} \times \overline{QP} sin\theta \tag{148}$$

Thus we can write

$$\begin{aligned} \vec{r}\,' &= \overline{OQ} + \overline{QP'} \\ &= \overline{OQ} + \overline{QP} cos\theta + \hat{\theta} \times \overline{QP} sin\theta \end{aligned} \tag{149}$$

Then using that

$$\overline{OQ} = (\vec{r} \cdot \hat{\theta})\hat{\theta} \tag{150}$$

so that

$$\overline{QP} = \vec{r} - \overline{OQ} = \vec{r} - (\vec{r} \cdot \hat{\theta})\hat{\theta} \tag{151}$$

we secure

$$\begin{aligned} \vec{r}\,' &= (\vec{r} \cdot \hat{\theta})\hat{\theta} + (\vec{r} cos\theta - (\vec{r} \cdot \hat{\theta})\hat{\theta} cos\theta) \\ &\quad + (\hat{\theta} \times \vec{r} sin\theta - (\vec{r} \cdot \hat{\theta})\hat{\theta} \times \hat{\theta} sin\theta) \\ &= \vec{r} cos\theta + (\vec{r} \cdot \hat{\theta})\hat{\theta}(1 - cos\theta) + \hat{\theta} \times \vec{r} sin\theta \end{aligned} \tag{152}$$

In terms of the components, this reads

$$x_i' = \Big[\delta_{ij} cos\theta + \hat{\theta}_i \hat{\theta}_j (1 - cos\theta) + \epsilon_{ikj} \hat{\theta}_k sin\theta\Big] x_j \tag{153}$$

from which we identify

$$R_{ij}(\vec{\theta}) = \delta_{ij} cos\theta + \hat{\theta}_i \hat{\theta}_j (1 - cos\theta) + \epsilon_{ikj} \hat{\theta}_k sin\theta \tag{154}$$

Consider the special case where $|\theta_i| << 1$ which corresponds to an infinitesimal rotation about the $\hat{\theta}$ axis. Retaining terms through linear in θ_i so that we can approximate $cos\theta \simeq 1$ and $sin\theta \simeq \theta$, we find

$$R_{ij}(\vec{\theta}) = \delta_{ij} + \theta \epsilon_{ikj} \hat{\theta}_k \tag{155}$$

Defining

$$\hat{\theta} \cdot \vec{t}_{ij} = i \epsilon_{ikj} \hat{\theta}_k \tag{156}$$

so that

$$(t_k)_{ij} = i \epsilon_{ikj} \tag{157}$$

Note that $(t_k)_{ij} = -(t_k)_{ji}$. Explicitly,

$$t_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \equiv t_x \tag{158}$$

$$t_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \equiv t_y \tag{159}$$

$$t_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv t_z \tag{160}$$

One can explicitly check that

$$[t_i, t_j] = i \epsilon_{ijk} t_k \tag{161}$$

which also follows since the ϵ_{ijk} satisfy the Jacobi identity

$$[[t_i, t_j], t_m] + [[t_m, t_i], t_j] + [[t_m, t_i], t_j] = 0 \tag{162}$$

so that

$$\epsilon_{ijk} \epsilon_{kmn} + \epsilon_{mik} \epsilon_{kjn} + \epsilon_{jmk} \epsilon_{kin} = 0 \tag{163}$$

Alternatively, we can define

$$\omega_{ij}(\vec{\theta}) = \theta \epsilon_{ikj} \hat{\theta}_k = -\omega_{ji}(\vec{\theta}) \tag{164}$$

so that for infinitesimal rotations

$$R_{ij}(\vec{\theta}) = \delta_{ij} + \omega_{ij}(\vec{\theta}) \tag{165}$$

Using the t_k matrices, we can rewrite $R(\vec{\theta})$ is a particularly compact form for finite rotations. To do this, introduce the combination

$$N_{ij} = -i(t_k)_{ij}\hat{\theta}_k = \epsilon_{ikj}\hat{\theta}_k \tag{166}$$

so that

$$\begin{aligned}
(N^2)_{ij} &= \hat{\theta}_i\hat{\theta}_j - \delta_{ij} \\
(N^3)_{ij} &= -N_{ij} \\
(N^4)_{ij} &= -(N^2)_{ij} \\
&\cdot \\
&\cdot \\
&\cdot \\
(N^{2n})_{ij} &= (-1)^{n+1}(N^2)_{ij} \ ; \ n \geq 1 \\
(N^{2n+1})_{ij} &= (-1)^n N_{ij} \ ; \ n \geq 0
\end{aligned} \tag{167}$$

Now consider

$$\begin{aligned}
(e^{\theta N})_{ij} &= \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!}\theta^n (N^n)_{ij} \\
&= \delta_{ij} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta^{2n+1}(N^{2n+1})_{ij} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\theta^{2n}(N^{2n})_{ij} \\
&= \delta_{ij} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta^{2n+1}(-1)^n N_{ij} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\theta^{2n}(-1)^{n+1}(N^2)_{ij} \\
&= \delta_{ij} + N_{ij}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} - (N^2)_{ij}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} \\
&= \delta_{ij} + N_{ij} sin\theta - (N^2)_{ij}(cos\theta - 1)
\end{aligned} \tag{168}$$

Recall that for finite rotations,

$$R_{ij}(\vec{\theta}) = \delta_{ij} cos\theta + \hat{\theta}_i\hat{\theta}_j(1 - cos\theta) + \epsilon_{ikj}\hat{\theta}_k sin\theta$$

$$
\begin{aligned}
&= \delta_{ij} cos\theta + (\hat{\theta}_i\hat{\theta}_j - \delta_{ij})(1 - cos\theta) + \delta_{ij}(1 - cos\theta) + N_{ij} sin\theta \\
&= \delta_{ij} + (N^2)_{ij}(1 - cos\theta) + N_{ij} sin\theta \\
&= (e^{\theta N})_{ij} \\
&= (e^{-i\vec{\theta}\cdot\vec{t}})_{ij}
\end{aligned} \tag{169}
$$

so that under a rotation of angle θ about the direction $\hat{\theta}$

$$
x'_i = R_{ij}(\vec{\theta})x_j = (e^{-i\vec{\theta}\cdot\vec{t}})_{ij}x_j \tag{170}
$$

with

$$
(t_k)_{ij} = i\epsilon_{ikj} \tag{171}
$$

The t_k form a matrix representation of the angular momentum algebra. Explicitly we can represent the operator J_k by the matrix

$$
J_k \Rightarrow \hbar t_k \tag{172}
$$

Thus the commutation relations

$$
[t_i, t_j] = i\epsilon_{ijk}t_k \tag{173}
$$

translates to the angular momentum operator algebra

$$
[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{174}
$$