8. For each of the following equivalences, determine if it is valid for all predicates P and Q. If yes then give a full explanation. If not then provide a counterexample.

   (a) $\exists x(P(x) \lor Q(x)) \;\equiv\; \exists x P(x) \lor \exists x Q(x)$

   Solution: *True.*

   If the LHS is *True* then there is a constant, say, a such that (i) $P(a) = T$, or (ii) $Q(a) = T$. In case (i) $\exists x P(x)$ is *True*, while in case (ii) $\exists x Q(x)$ is *True*, so that in both cases the RHS is *True*,

   If the RHS is *True*, then (i) there is a constant, say, a such that $P(a) = T$, so that the LHS is also True with $x = a$, or (ii) there is a constant, say, b such that $Q(b) = T$, so that the LHS is also True with $x = b$.

   (b) $\exists x(P(x) \land Q(x)) \;\equiv\; \exists x P(x) \land \exists x Q(x)$

   Solution: *False.*

   Suppose $P(a)$ is *True* for some constant a, and *False* for all constants $b \neq a$. Suppose also $Q(b)$ is *True* for some constant b, and *False* for all constants $a \neq b$. Then RHS is *True* and LHS is *False*.

   (c) $\forall x \forall y(P(x) \land Q(y)) \;\equiv\; \forall x P(x) \land \forall x Q(x)$

   Solution: *True.*

   If the RHS is *True* then P is always *True* and Q is always *True*. Then the LHS is also always *True*.

   If the RHS is *False* then there is a constant, say a, such that $P(a) = F$, or there is a constant, say b, such that $Q(b) = F$. In the first case $P(a) \land Q(a)$ is *False*, and in the second case $P(b) \land Q(b)$ is *False*, so in any case the LHS is also *False*.

   (d) $\forall x \forall y(P(x) \lor Q(y)) \;\equiv\; \forall x P(x) \lor \forall y Q(y)$

   Solution: *True.*

   If the RHS is *True* then we must consider two cases: (i): P is always True or (ii) Q is always True. In either case the LHS is True also.

   If on the other hand the RHS is *False* then there is a constant, say a, such that $P(a) = F$, and there is a constant, say b, such that $Q(b) = F$. Then the LHS is seen to be False also, namely by picking $x = a$ and $y = b$.