8300 Day One Fall 2003. MWF 10:10 am. room 410.
Introduction to algebraic geometry: some fundamental problems, and connections between algebra and topology

Basic Questions:
What do solution sets of polynomial equations "look like"?
a. Are the solution sets empty or non empty?
b. If there are solutions, when are there infinitely many?
c. When can we parametrize the infinite solution set?
d. When are there a finite number of solutions, and then how many are there?

Since one application of algebraic geometry of interest to many of us, is to number theory, we look at a few examples of how geometry, as well as topology and analysis can impact number theory. If f is a polynomial in one or more variables, with integral or rational coefficients, we can look for integral, rational, real, or complex solutions of f(X) = 0. The complex solution set has a lot of classical geometry, for instance it is a CW complex and usually a complex manifold, while the set of integral or rational solutions belong more to number theory. The beautiful phenomenon is that there are relations between these different sets of solutions. E.g. the geometry of the set of complex solutions can affect the nature of the rational solutions. Here is a simple but important example.

Rational parametrizations:
Recall that in linear algebra we can always parametrize the solutions of a system of linear equations, i.e. there is a linear map from a linear parameter space onto the set of solutions, so that any choice of parameter yields a solution of the system of equations. Suppose we could do that for the solutions of a polynomial equation. I.e. suppose f(X,Y) has rational coefficients and there are non constant rational functions x(t), y(t) with rational coefficients, such that for all t, f(x(t),y(t)) = 0. Then we could produce as many rational solutions of f(X,Y) = 0 as we like, since for any rational value of t, then (x(t),y(t)) is a rational solution of f(X,Y) = 0. It turns out this sort of parametrization is usually not possible for purely topological reasons.

I.e. for each irreducible polynomial f(X,Y), the associated complex solutions of {f(X,Y) = 0} form a topological space which is a complex one manifold except at a finite set of points. One can then modify this space, by compactifying it with the addition of a finite number of points, then desingularize it by doing a simple surgery at another finite set of points, and obtain a complex one manifold called the associated Riemann surface of the given polynomial. The assignment of a Riemann surface to an irreducible plane curve is a "functor", i.e. it is natural in the sense that a rational map between two curves induces a holomorphic map of their Riemann surfaces. Hence a rational parametrization of a curve f(X,Y) = 0 over Q, which is a rational map from the rational line Q to the curve {f=0}, yields also a rational map from C to the complex points of f = 0, and a holomorphic map from the Riemann surface of Q to the Riemann surface of {f=0}. Now the Riemann surface of Q is the one point compactification of C, namely the Riemann sphere.

So a rational parametrization of {f=0} yields a surjective holomorphic map from the Riemann sphere to the Riemann surface of {f=0}, which is some compact topological surface, with a complex structure. Now a surjective holomorphic map allows us to pull back non zero differential forms from the target, to non zero differential forms on the source, but the Riemann sphere has no nonzero differential forms. So this holomorphic map cannot exist if the Riemann surface of {f=0} has any non zero holomorphic forms. The existence of such forms on a

Riemann surface is equivalent to the Euler characteristic $ç_{top}$ being ≤ 0, so a rational parametrization cannot exist for a curve whose Riemann surface is a compact surface of genus $g = 1 - (1/2)ç_{top} \geq 1$.

Intuitively, a surface is a doughnut possibly with holes, the genus is the number of holes, and this number can be seen from a model of the curve made from lines. I.e. the genus, being an integer, is invariant under deformation, so to compute it we may assume our curve is a union of lines. A triangle obviously has one hole hence we claim a "non singular" curve of degree three has genus one, whereas a union of 4 lines has three holes, hence a non singular curve of degree 4 has genus 3, etc. Thus if the degree of the curve is ≥ 3, and there are no singular points of either the complex curve or its projectivization (compactification), then the genus is ≥ 1, so there can be no rational parametrization.

We can compute the genus without using this deformation argument, by projecting our surface onto the complex x - axis in the x,y plane and computing the branching behavior. For example the cubic curve $y^2 = x(x-1)(x+1)$ projects 2:1 onto the x axis, branched over the three points $x = 0,1,-1$. I.e. over each of these three points, there is only one preimage point instead of two. This projection induces a holomorphic map of the associated Riemann surfaces as before.

The (compact) Riemann surface of the x axis is the one point compactification of C, namely the Riemann sphere, and the Riemann surface of the curve $y^2 = x(x-1)(x+1)$ is obtained by adding either one or two points over the point at infinity on the Riemann sphere. If it were the case that we add two points then neither is a branch point, but it follows from Hurwitz' formula that such a map of compact surfaces must have an even number of branch points, so we in fact must add one point at infinity.

This gives genus one for the curve $y^2 = x(x-1)(x+1)$ as follows. Recall that the Euler characteristic of a compact surface is equal to V-E+F, where V is the number of vertices, E the number of edges, and F the number of faces in a triangulation of the surface. If we triangulate the Riemann sphere into small triangles and make sure each branch point is a vertex, then a branched 2:1 cover pulls back each triangle on the sphere to two triangles on the curve $y^2 = x(x-1)(x+1)$, and each edge to two edges, and each vertex either to two vertices, or only one if the vertex is a branch point.

Thus, if V,E,F are the numbers associated to our triangulation of the sphere, then the Euler characteristic of the Riemann surface of the curve $y^2 = x(x-1)(x+1)$, equals $(2V-b) - 2E + 2F = 2(V-E+F) - b$. Since for the sphere $V-E+F = 2$, this gives 0 for the Euler characteristic of the curve $y^2 = x(x-1)(x+1)$, which thus has genus 1.

The computation shows why there is always an even number of branch points for such a map as well, since the branch order is the difference between the Euler characteristic of the source, and a multiple of the Euler characteristic of the target, and the difference of these even numbers must be even.

Notice in this calculation that we are using a map from our curve onto a sphere to prove there can be no map from a sphere onto our curve. Given that parametrization maps do not often exist, the existence of these maps in the other direction is a fundamental tool for studying algebraic sets. I.e. we will show fairly early that there are always "finite" maps from any affine algebraic set of dimension n onto the affine space k^n of dimension n, and eventually show that rational maps in the other direction, i.e. from affine space onto our algebraic set, exist only in special cases.

Guided by this geometric, topological, and analytical reasoning, we may conjecture that a plane curve of degree ≥ 3 over any field k, cannot be rationally parametrized if there are no "singular" points of the given curve over the algebraic closure of k. This is in fact true, and can be proved in a completely algebraic way, involving an algebraic version of differential forms, namely formal derivatives. We give a direct proof in the notes from 8300 in Fall 2001 (for the Fermat cubic in characteristic $\neq 2, 3$?) as follows:

Non rationality of Fermat cubic curve

Assume that $(x/z)^3 + (y/z)^3 = 1$, where x, y,z are polynomial functions of t with no common factors. (since neither x/z nor y/z is constant, neither x nor y is zero.) Multiply through to obtain (1): $x^3 + y^3 = z^3$, and differentiate to obtain (2): $x^2x' + y^2y' = z^2z'$, where ' denotes differentiation w.r.t. t. Now we want to eliminate the z terms. multiply (1) by z' and (2) by z, and subtract, to obtain $z'x^3 + z'y^3 = x'x^2z + y'y^2z$. Collecting terms, and factoring gives $x^2(xz'-zx') = y^2(y'z - yz')$. If either $(y'z - yz')=0$, or $(xz'-zx') = 0$, then both do, and hence by the quotient rule for derivatives, we would have x/z and y/z constant. Since x,y are rel prime, x^2 divides $(yz'-zy')$, in C[t], and thus $2\text{degree}(x) \leq \deg(y)+\deg(z)-1$. Repeating the argument for each of the other two variables, i.e. eliminating the x and y terms, leads to the same inequality with the variables permuted. Adding the 3 inequalities gives $2[\deg(x)+\deg(y)+\deg(z)] \leq 2[\deg(x)+\deg(y)+\deg(z)] - 3$, a contradiction. QED.

Thus this result is true over Q and C, for topological and analytic reasons, and the topological and analytic invariants involved have algebraic incarnations, which lead to a conceptual proof in characteristic $p > 0$ as well, as we will see later. I.e. we will give a purely algebraic definition of the genus, and prove that a non constant rational map cannot exist if the target has larger genus than the source.

There are also sufficient criteria for existence of parametrizations, based on topological criteria as well. I.e. if the Riemann surface of a curve has genus zero then the curve has a rational parametrization over C, but in some cases the parametrization also exists over Q. Now a curve with a rational parametrization has infinitely many rational points, so the curve $X^2 + Y^2 + 1 = 0$, which has no real points at all, hence no rational points, cannot be rationally parametrizable over Q. In fact the complex points of this curve do have genus zero, so there is a parametrization over C. To get a parametrization we need at least one rational point.

Then we can prove that an irreducible plane curve of degree 4 with four rational points, three of which are singular on the associated Riemann surface, is parametrizable over Q. I.e. a smooth curve of degree 4 has genus 3, and three singular points imply that three homology cycles on the riemann surface have been shrunk each to a point, collapsing the three holes and resulting in a Riemann surface of genus zero.

The fourth rational point then allows the parametrization. I.e. we consider all conics passing through all 4 rational points, and a general one of these meets the curve in one further point, for which we can solve rationally in terms of the others and the coefficients of the two intersecting curves. Hence this residual intersection point is also rational. Considered as a rational map from the quartic to a plane curve, the target is a conic, and the 4th points maps to a rational point of this conic.

Then the result follows from the fact that a conic with one rational point is parametrizable. In particular, it has an infinite number of rational points. technically we have found a map of degree one from our curve to a parametrizable curve, but we can also use conics to map the parametrizable curve to our curve, thus obtaining a parametrization of our curve.

In fact with suitable coordinates, both the map from our quartic to the parametrizable conic and the inverse map can be given by the same map, the standard quadratic transform (yz, xz, xy). It should follow in a similar way, that an irreducible curve of degree d with "d-1 choose 2" rational singular points, and one more rational point, at least in characteristic zero, is rationally parametrizable.

Now that we know the method of parametrization is inapplicable to smooth curves of genus ≥ 1, we can ask whether there is some other reason for such curves over Q to have an infinite number of rational points. For genus one, i.e. smooth plane curves of degree 3, there is a method of producing more rational points from a given one, the "tangent method". I.e. the tangent line at a rational point intersects the curve in a second rational point. Then the tangent line at the second point meets the curve again at another rational point, etc...

So it is conceivable that a curve of genus one may have an infinite number of rational points, and in fact this can happen. Mordell conjectured that if the genus is ≥ 2 however, then in fact there are never an infinite number of rational points, and this was proved by Deligne about 30 years ago.

As foreshadowed by the arguments above, some of our principal goals will be to define an irreducible variety (the geometric analog of irreducible polynomials), define dimension of an irreducible variety, then show that every irreducible n dimensional affine or projective variety can be mapped finitely onto either an n dimensional affine or an n dimensional projective space, and then develop the algebraic analog of differential calculus and use it to prove that a smooth n dimensional projective hypersurface of degree $\geq$ n+2 cannot be rationally parametrized.

Along the way we will discuss the geometry of some hypersurfaces of degree ≤ 3, in the plane, in 3 space, and in 4 space. We will also discuss the case of a non singular quadric in 5 space, which parametrizes the famous "Grassman variety", the parameter space or "moduli" variety for lines in three space. There are still many difficult open questions about which varieties have rational parametrizations, especially which varieties admit parametrizations of degree one. For example, a 2:1 rational parametrization of any non singular cubic threefold in 4 space was known classically, and it was suspected that parametrization of no degree one could exist. The proof of this non existence result was achieved in 1972 using deep and beautiful results from the theory of abelian varieties.

Outline of Topics

We will describe the most basic tools and concepts of algebraic geometry, in roughly the order of topics below. I want to introduce enough language to state the important Riemann Roch problem, and to understand the ingredients in the statement of its (partial) solution, the Riemann Roch theorem. The proof of the Riemann Roch theorem is best done with sheaf cohomology, the topic of another course. The importance of the Riemann Roch theorem cannot be overstated. I will say one thing about it. If we want to classify all the projective algebraic varieties in the world, there is a standard approach:

1) classify all abstract algebraic varieties,
2) for each abstract variety, determine all its projective embeddings.

The Riemann Roch theorem is the primary tool in step 2), which can be studied for a given variety independently of step 1).

My goal is to cover at least parts I-V of the outline below, i.e. to prove Bezout's theorem.

Outline

I. Algebraic sets

decomposition into irreducible components.
the dimension of an irreducible algebraic set.
the differences between affine and projective sets.

Dimension of intersections
II. Algebraic maps
finite maps
Veronese and Segre maps
universal finite maps (normalization)
closedness of all maps on projective sets
birational equivalence
III. Nonsingularity
Zariski tangent spaces
Unique factorization in the local ring of a non singular point
Local equations for subvarieties
IV. Divisors
Weil divisors vs Cartier divisors, linear equivalence
A principal divisor on a curve has degree zero
Bezout theorem for curves, applications to rationality of curves
The role of divisors in describing maps to projective space
The Riemann Roch problem (compute the dimension of the space L(D) of a divisor)
V. Intersection numbers on non singular varieties
Definition of intersection numbers of divisors in general position
bilinearity of intersection product
Invariance under linear equivalence
Moving divisors up to linear equivalence
Definition of intersection numbers of divisors not in general position
Bezout's theorem
VI. Differentials
Rational and regular differential forms
The canonical divisor class K on a non singular variety
Statement of Riemann Roch for n.s. curves: ç(D) - ç(0) = deg(D),
Statement of Riemann Roch for n.s. surfaces:

2(ç(D) - ç(0)) = D\(D-K); and (over ^) 12\ç(O) = K^2 + ç$_{top}$

VII Birational maps
Blowing up a point
Birational maps of surfaces

Remarks:

This is a course about algebraic varieties, not schemes, but schemes are unavoidable even here, and we will learn a few things about subschemes of classical varieties, especially when doing intersection theory. For us a scheme is an algebraic set plus a distinguished ideal of defining equations (among the infinitely many choices).

Along the way we will learn the geometric meaning of a lot of concepts from algebra, such as divides (contains), product (union), domain (irreducible set), primary decomposition (decomposition into irreducible components), non minimal primes (embedded subvarieties), zero divisors (existence of more than one component), prime ideal (irreducible subset), maximal ideal (point), transcendence degree (dimension), regular sequences (flag of subvarieties of decreasing dimension), unique factorization (codimension one sets have one defining equation), integral closure (local irreducibility), length (intersection multiplicity), integral ring extension (proper map with finite fibers), field isomorphism (isomorphism of dense open sets), ring surjection (closed embedding), ring injection (dense morphism), homogeneous polynomials (compact sets), Nakayama lemma (implicit function theorem), radical ideal (algebraic set), ideal ("scheme", e.g. alg set plus multiplicities).

We will work over an algebraically closed field of arbitrary characteristic. The geometric notions mirror those studied in differential topology but the treatment is completely algebraic, to serve the needs of algebraists and number theorists.

Prerequisites:

You will need to know something about noetherian rings and ideals, fields, vector spaces, polynomials, and "localization" of rings (generalizing how to construct a field of fractions from a domain). We will always state facts we need, especially if you ask, so you can survive without full knowledge of them. (We will not use the theory of primary decomposition.)

The difference between a scheme and a variety

In the affine case, i.e. as sub objects of k^n determined by ideals in $k[T_1,...,T_n]$, to ask the difference between the scheme and the variety determined by an ideal I is to ask the question: how much more information is contained in I as opposed to its radical rad(I)? Just as the ideal I determines its radical, the scheme "spec(k[T]/I)" determines the variety V(I). The variety is just the information contained in the underlying point set of the scheme, and the two ideals I and rad(I) determine exactly the same point set in k^n.

The question to begin with is the simpler one, how much information is contained in the variety, i.e. in rad(I)? For example, the ideal rad(I) is uniquely the intersection of the minimal prime ideals $P_1,...,P_r$ containing I. Thus rad(I) determines these prime ideals, which correspond to the irreducible components of the variety V(I). However the full ideal I may determine other prime ideals. If we choose any irredundant primary decomposition of I = Q_1€....Q_{r+s}, such that the prime ideals rad(Q_i) = P_i are minimal for i =1,...,s, then from the theory of primary decomposition one learns that I determines the primary ideals $Q_1,...,Q_s$, hence also the prime ideals $P_1,...,P_s$, as well as the prime ideals $P_{s+1},...,P_{s+t}$.

Thus I determines

(1) the irreducible decomposition of the point set V(I) = V(P_1)⁄....⁄V(P_s)

(2) the "embedded" subvarieties V(P_{s+1}),....,V(P_{s+t}), each properly contained in some V(P_i) with $i \leq s$.

(3) the primary ideals $Q_1,....,Q_s$ with V(Q_i) = V(P_i) for i=1,...,s.

The radical rad(I) contains only the information in (1). The embedded scheme determined by I is equivalent to the pair I fi k[T], while the abstract scheme determined by I is equivalent to the quotient ring k[T]/I. Similarly the embedded variety determined by I is equivalent to the pair rad(I) fi k[T], and the abstract variety determined by I is equivalent to the quotient ring k[T]/rad(I).

In this course we will tend to ignore the information in (2), or consider only examples where the ideals $Q_{s+1},...,Q_{s+t}$ do not exist, i.e. we will not study schemes with embedded components. In some of our work however, showing these components do not exist will be a principal concern for us. But even for each ordinary component V(P_i), i=1...s, of the variety, the ideal Q_i determined by I, carries extra information about that component. We will want to use this information in several settings. What is that information? The simplest piece of data we can derive from it is an integer, the multiplicity of the component. Thus a scheme determines a cycle, an integer associated to each irreducible component of the associated variety.

To see how this may be defined, consider the quotient ring k[T]/I, which kills off all ideals smaller than I, by setting them equal to zero. Then localize this ring at one of the prime ideals P_i, which kills off all ideals not contained in P_i, by making them into the unit ideal. There is now only one prime ideal left, P which is now both minimal and maximal. In this localized zero dimensional ring $(k[T]/I)_P$, there is now only the prime ideal P, and I is an ideal such that rad(I) = P. Then asking what is the difference between the information about the component V(P) which is contained in just the variety V(I), and the fuller information about that component which is contained in the scheme determined by I, is the same as asking the difference between the information in the field $(k[T]/P)_P = (k[T]/\mathrm{rad}(I))_P$, and the zero dimensional ring $(k[T]/I)_P$.

Now the ring k[T]/P is the affine ring of regular functions which is equivalent to the irreducible component V(P), and the field $(k[T]/P)_P$ is the field of rational functions on this component. The slightly larger ring $(k[T]/I)_P$ constructed from the full ideal I, determines that field of rational functions too, but is a little bit larger ring since it has some non zero ideals. Just as we measure the dimension of a ring by its chains of prime ideals, we can measure the "size" of a zero dimensional ring by using chains of arbitrary ideals, the "length" of the ring. I.e. $(k[T]/I)_P$ is a ring whose maximal ideal is nilpotent, and it follows that it has finite length. This length assigns an integer to the component, that measures the multiplicity with which it should be counted in some problems such as intersection multiplicity.

Detailed Outline

In **chapter I**, we study the set theoretic properties of varieties, their components and the dimensions of the components, for varieties and for fibers of morphisms between varieties. The only time scheme theoretic properties come up is when we briefly mention the degree of a projective hypersurface, to distinguish the hypersurfaces V(T) and $V(T^r)$, which have the same point set but different degrees.

In **chapter II**, we study the concepts of tangent space and tangent cone to a variety at a point. The tangent cone to a variety of codimension one in affine space, is a hypersurface which approximates the variety well at the point, and is dedfined by the lowest order term of the Taylor series for the variety at the given point. Since it is defined by a homogeneous polynomial which may not be square free, it may not be a variety, and in fact is best considered as a scheme since as a projective hypersurface it has a degree. E.g. the variety $y-x^3$ has tangent cone at (0,0) defined by the ideal (y), and the variety y^2-x^3 has tangent cone defined by (y^2). Both tangent cones have the same point set, the x axis, but the second one if considered as a scheme, or a cycle, would allow us to associate the mutliplicity 2 to the point (0,0) on the curve y^2-x^3. The treatment in chapter II ignores this and analyzes only the point set of a general tangent cone, so does not introduce scheme methods. (Early in chapter 1 however, the notion of multiplicity is introduced in the case of plane curves like these.)

The topic of birational transforms in section 4 may have to be skipped for lack of time, although it is very important and basic. This is a standard construction. Given a smooth n dimensional variety Y, we construct another smooth n dimensional variety X and a morphism ß:X¨Y, such the map ß is an isomorphism over all but one one point of Y, and over that point the fiber is $@^{n-1}$. This is called blowing up that one point. We will need the concept of normalization of a variety in section 5, especially that of a projective curve. We may have to skip all but the definition and statement of basic properties of the degree of a map in section 6.

In **chapter III**, we begin to study the simplest schemes, locally principal subschemes of smooth varieties, called "effective divisors". These are essentially finite sets of irreducible subvarieties of codimension one, each counted with a non negative integer multiplicity. In projective space these are just exactly hypersurfaces, considered with multiplicities so as to have a well defined degree. In the notation given above for studying the information provided about a component by an ideal, when the component has codimension one, we have a local ring $(k[T]/I)_P$ in which the maximal ideal is both principal and nilpotent, and the degree of nilpotency will equal the length of the ring. I.e. the maximal ideal is generated by some element f, and the ideal of the component has a local equation of form f^r for some integer r, and that r assigns the multiplicity we want to the component.

Each rational function has an associated divisor which determines that function on a projective variety up to a constant multiple. We prove the important and basic theorem that the divisor of a rational function on a smooth projective curve has degree zero, which lies at the basis of our main result on intersection numbers in chapter IV.

Modding out the group of all divisors by the subgroup of divisors of functions, gives an important invariant of a variety X, called its Picard variety. Although the group of divisors is an infinite dimensional free group, this quotient group Pic(X) is a disjoint union of finite dimensional algebraic varieties itself. Determining the dimension of this variety is important as is the problem of determining all effective divisors which are equivalent in Pic(X), i.e. which effective divisors are linearly equivalent to others.

The problem of determining the dimension of the projective space |D| of effective divisors in the linear equivalence class containing a given divisor D, is called the Riemann Roch problem. Every divisor on X for which this dimension is $r \geq 1$, determines a rational map from X to @^{r}, and these projective mappings are very useful in studying the geometry of X. So divisor classes let us study the relation between the intrinsic geometry of X and the geometry of its embedded realizations in projective space.

It turns out one can say something about the dimension of this space |D| in terms of the ``intersection numbers`` (see chapter IV) of D and those in a very important and intrinsic divisor class, the ``canonical`` divisor class |K|. Canonical divisors have many important applications, e.g. to non rationality of hypersurfaces, and are studied in chapter III. For instance the fact that the canonical divisor of a smooth plane cubic is 0 implies such a cubic is not rational or parametrizable.

In **chapter IV**, we consider more general local complete intersection schemes, still on a smooth variety. If X is n dimensional and smooth, consider $r \geq n$ effective divisors $D_1,....,D_r$ on X, defined by local equations $f_1,....,f_r$, and whose intersection D_1€....€D_r = Y has pure dimension n-r. Then it follows that the subscheme Y has no embedded components, i.e. the ideal $I = (f_1,....,f_r)$ has no embedded primes. These "local complete intersection schemes" generalize divisors in the sense that the codimension still equals the number of local equations needed to define the ideal.

In this case the main information about the components of Y contained in I is their multiplicities, defined by the lengths of the local rings discussed above. When r = n, and Y is a finite set of points, we will show the local intersection multiplicites assigned in this way to the points of the set Y = D_1€....€D_r, give numbers which satisfy Bezout's theorem. I.e. in case C is projective space we will prove that the sum of these local intersection multiplicities equals the product of the degrees of the hypersurfaces D_i. In particular, n hypersurfaces in @^{n} of degrees $d_1,....,d_n$ cannot have more than °d_i common points, unless they have infinitely many. This is not true over a non algebraically closed field like the reals, where n real hypersurfaces can have more common real points than the product of their degrees, and still only a finite number of real points in common. The explanation lies in the fact that such counterexamples to Bezout over %, have infinitely common points over ^. Bezout`s theorem has some lovely and striking applications, e.g. to division algebras over %.

More generally, we prove ``Bezout`` on any smooth variety, that intersection numbers are invariant under changing the divisors to linearly equivalent ones. With this background we not only prove Bezout in $@^n$, but state and apply the famous ``Riemann Roch`` theorem for curves and surfaces. This is a formula relating the dimension of |D|, a problem posed in chapter III, to the intersection numbers of D and a canonical divisor K.

Of course in this introduction to schemes, we have only been discussing schemes which are finitely generated over an algebraically closed field. In general scheme theory, one allows not only that the ring has nilpotents, but the ring need not be a quotient of a polynomial ring over a field, much less an algebraically closed one. One simply considers a ring, possibly noetherian, and asks how the algebraic constructions which we made for the more classical setting, behave now. This allows the possibility of using geometric intuition in the analysis of general rings. Although one can make many of the definitions, one cannot prove many theorems in this generality. Still it is fruitful in some settings. For example in the non commutative setting, Professor Jon Carlson was able to show that if certain non commutative rings studied in representation theory had no zero divisors, then the associated representation varieties were irreducible, generalizing a standard fact in commutative classical algebraic geometry.

Remarks on the Basic Questions:

1. What do the solution sets of polynomial equations "look like"? I.e. what are their geometric properties?

a. Are the solution sets empty or non empty?

This is often very difficult over #, $, (see Silverman - Tate for examples) even over % but here we can at least use IVT;

<u>Value of working over ^:</u> nullstellensatz says they are always non empty except when the equations generate the unit ideal. e.g. x-1 and x-2 generate unit ideal since their difference is 1. This is entirely analogous to the criterion for existence of solutions of linear equations, a single non homogeneous equation has solutions, and a non homogeneous linear system has solutions if and only if we cannot reduce to an equation of form 0 = 1 by Gaussian elimination, i.e. if and only if the equations do not generate the unit ideal. In fact here it is sufficient to use linear combinations of the equations to generate the unit from an inconsistent system. In several variables again, one non - constant non homogeneous equation always has solutions, either by the nullsatz or directly by induction on the number of variables.

<u>Advantage of working in "projective space",</u> i.e. with homogeneous equations: the results which hold for homogeneous linear equations generalize: not only do all homogeneous systems of equations of any degree have the zero solution, but systems of fewer than n homogeneous equations in n variables always have non zero solutions, again over an algebraically closed field (otherwise x^2+y^2 does not, over % for instance.) Starting from non homogeneous equations and homogenizing them yields solutions "at infinity", e.g. the parallel lines in the affine x,y plane x-1 and x-2 become the lines x-z and x-2z in the projective x,y,z plane, which have the common solution (0,1,0) lying on the line at infinity z=0. In general homogeneous equations provide solutions which compactify the solutions of non homogeneous equations.

b. <u>If there are solutions, when are there infinitely many?</u>

always assume we are working over ^ or an algebraically closed field, from now on. Assuming solutions exist, if there are more variables than equations, then there should be an infinite number of solutions. In particular, a system of fewer than n homogeneous equations in n+1 variables have an infinite number of non zero, non proportional, solutions. For example, one homogeneous equation in three variables, i.e. one equation in the projective plane, over an algebraically closed field, should have an infinite number of solutions.

What does that infinite set look like? Over ^, we can try to use calculus, i.e. the implicit function theorem. Thus, assume the derivative of an affine form f(x,y) of the equation at some solution is non zero, i.e. at least one partial is not zero, e.g. Δf/Δy ± 0 at p = (a,b). Then the IFT says that near p, the solution set {f=0} is the graph of an analytic function y(x) with y(a) = b. In particular, the part of the solution set near p is analytically isomorphic to an open disc in ^. Thus if at each point of the solution set, at least one partial in an affine equation is non zero, then the solution set is everywhere locally isomorphic to such a disk, i,e, the solution set is a one dimensional complex manifold, and thus also a compact oriented real 2-manifold. Since it can be shown that it is connected, it follows from topology that the solution set is homeomorphic to a sphere with $g \geq 0$ handles attached, where the number of handles g is called the genus. For this reason we call such a solution set a (complex) curve, or Riemann surface. But what is the genus? One way to tell: degeneration to a union of a general family of lines. Another way: give a branched covering map from the surface to a line, i.e. a topological sphere, and try to count the branching number. Then use Hurwitz formula.

c. When can we parametrize the infinite solution set?

Try to understand a solution set by parametrizing it, i.e. by finding a map onto it, or densely onto it, from affine space, e.g. from $!^1$ onto a curve. When is this possible? From our topological analysis, we might think only when the genus is zero. At least it follows from complex analysis that this is a necessary condition but why is it sufficient? What are some examples of this phenomenon? A line, x+y = 1, or a conic such as $x^2+y^2 = 1$. We can parametrize these, by projection, parametrize the line by projecting from a point off the line, and parametrize the conic by projecting from a point on the conic. Then consider the special cubics $y^2 - x^3$, or $y^2 - x^3 - x^2$, in which cases one must project from the unique singular point of the cubic. A quartic with three ordinary double points at the standard vertices can be parametrized by a conic, using the standard quadratic transformation.

Try to generate a conjecture on when an irreducible plane curve is rational, in terms of its number of singular points. Look at some reducible examples, such as a cubic composed of a conic and a line, which does not admit a dense map from one copy of a line. Also note a smooth cubic plus a line is a quartic with three ordinary double points but also does not admit a dense parametrization from a line. A sextic with 9 ordinary double points has genus one, but a sextic composed of two cubics can also have 9 ordinary double points but is a union of two curves each of genus one. Could we also have a parametrizable quartic with only two double points? A curves of form $y^2 = f(x)$ where f has degree 4 in x, and two equal roots, seems to be parametrizable and only to have one (finite) singular point, so one is motivated to introduce a measure of badness of the singularity, such as multiplicity, and to look at points at infinity. With homogeneous coords, e.g. $y^2 = x^4+x^2$, becomes $y^2z^2 = x^4+x^2z^2$ which has a point on the line z = 0, at (0,1,0), i.e. at (0,0) with affine equation (set y=1) $z^2 = x^4+x^2z^2$, a non ordinary double point.

d. When are there a finite number of solutions, and then how many are there? A general system of n homogeneous equations in n+1 variables should have a finite number of common solutions, how many? Bezout gives the answer as follows. The number of common points should be either infinite or not more than the product of the degees of the equations. Consider the case of linear equations, then it is one. Consider two plane curves. Try to prove Bezout in case one of the curves is parametrizable, such as a line or smooth conic.

The number of points in a finite solution set generalizes to the question of the number of ``components`` for an arbitrary solution set. I.e. there is a notion of irreducible components, and for each component, a notion of dimension of that component.