

A Software Application For Simulating The Effects Of Space Charge In A Rod-Plane Gap

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1 Governing Problem Statement: Total Field Computation

1.1 Calculating The Total Field ($\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_{sp} + \vec{E}_{I_{rod}} + \vec{E}_{I_{plane}}$)

The primary objective of this simulation is to compute the total, instantaneous electric field distribution $\vec{E}_{total}$ within a highly divergent rod-plane gap during a corona discharge event. The presence of drifting space charge fundamentally alters the geometric field, requiring a coupled **electrostatic solution**.

To determine the net electric field at any point in space, we employ the principle of linear superposition, modeling the total field as a vector sum of four distinct components:

$$\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_{sp} + \vec{E}_{I_{rod}} + \vec{E}_{I_{plane}} \quad (1)$$

where:

- $\vec{E}_{ext}$: Is the external (geometric) electric field caused solely by the applied voltage on the electrodes in a completely uncharged medium.
- $\vec{E}_{sp}$: Is the space charge electric field generated directly by the cloud of drifting ions in the inter-electrode gap.
- $\vec{E}_{I_{rod}}$: Is the image charge field accounting for the boundary conditions on the curved surface of the rod-electrode (ensuring the total potential on the conductor matches the boundary values).
- $\vec{E}_{I_{plane}}$: Is the image charge field enforcing the zero-potential boundary condition on the grounded plane-electrode.

1.2 Foundational Simplifications and Physical Assumptions

To maintain computational feasibility while preserving physical accuracy, the numerical model operates under three core physical simplifications regarding the dielectric medium (air):

- **Electrostatic Linearity**

By expressing $\vec{E}_{total}$ as a straightforward algebraic sum of its components, we assume the governing medium behaves linearly. This implies that the presence of space charge does not alter the fundamental permittivity of the medium, allowing individual fields to exist independently and sum together without cross-modulation or non-linear coupling effects.

- **Constant Relative Permittivity (ϵ_r)**

We assume that the relative permittivity of the air medium is perfectly constant and uniform throughout the entire gap ($\epsilon_r \approx 1$). In a real corona discharge, the intense localized electric field creates a highly thermalized ionization zone (plasma sheath) where localized temperature spikes, gas density drops, and electron-ion densities could technically alter the local dielectric constant. For this kinematics model, these micro-scale fluctuations are assumed to have a negligible impact on the macro-scale field distribution.

- **Isotropic and Homogeneous Medium**

The medium is modeled as isotropic and homogeneous. This means its physical properties, specifically its permittivity and ion mobility are identical in all directions and at all spatial coordinates. The electric displacement vector $\vec{D}$ remains perfectly parallel to the electric field vector $\vec{E}$ via: $\vec{D} = \epsilon_r \epsilon_0 \vec{E}$. This assumption ensures that the field's direction depends purely on the spatial layout of the charges and electrodes, rather than any directional preferences or crystal-like biases in the gas medium itself.

2 Theoretical Analysis

2.1 Space Charge Density Function ($\rho(r)$)

In order to calculate the **Total Electric Field distribution** due to Space Charge (and its images), we first need a function that describes the space charge density as a function of space. Since this Thesis is based on the work of **Wang Wenduan, Wang Haiting** “A New Model For Simulating The Effects Of Space Charge In Gas Discharge” [1], we will use their improved model, that is given by:

$$\rho(r) = \frac{A}{(r + C)^B} \left[\frac{C}{m^3} \right] \quad (2)$$

where:

- $A[C \cdot m^{B-3}]$: **Peak Charge Density / Scaling Amplitude**

This coefficient governs the overall intensity or scale of the space charge distribution. Physically, when evaluating the system at the absolute boundary or origin line ($r \rightarrow 0$), **A** acts as the baseline factor that determines the maximum localized charge density in tandem with **C**.

- $B[\text{Dimensionless}]$: **Spatial Decay Exponent / Attenuation Rate**

This parameter dictates how aggressively the space charge density drops off as you move radially outward from the discharge source. A higher value of **B** represents a highly localized, rapidly decaying plasma sheath or ionization zone, whereas a lower value indicates a more dispersed, long-range space charge cloud.

- $C[m]$: **Spatial Core Radius / Boundary Singularity Regulator**

This is a spatial offset parameter that prevents physical and mathematical singularities at the inner boundary. By ensuring that $\lim_{r \rightarrow 0} \rho(r) = \frac{A}{C^B}$, it models the finite physical limits of the electrode or the core of the discharge zone, keeping the initial charge density stable and finite instead of letting it approach infinity [1].

- $r[m]$: **Radial Distance / Kinematic Coordinate**

The distance measured radially outward from the apex (tip) of the rod-electrode into the inter-electrode gap. It serves as the primary spatial coordinate for tracking the kinematics and localized attenuation of the drifting ions or space charge.

2.2 A Note On The Geometric Field ($\vec{E}_{ext}$) & The Plane Image Field ($\vec{E}_{I_{plane}}$)

In this chapter, we are focused on the calculation of the total electric field distribution due to **space charge**, $\vec{E}_{sp,total} = \vec{E}_{sp} + \vec{E}_{I_{rod}} + \vec{E}_{I_{plane}}$. The calculation of the external (applied) geometric field $\vec{E}_{ext}$, will be described later using the **Charge Simulation Method** [2]. Also, its important to mention that this Thesis is based on the work of **Wang Wenduan, Wang Haiting** “A New Model For Simulating The Effects Of Space Charge In Gas Discharge” [1], where they assume $\vec{E}_{I_{plane}} \approx 0$, since they study the space charge effects **only on the region close to the rod tip**.

2.3 Calculating The Space Charge Electric Field ($\vec{E}_{sp}$)

In figure 1, the minimum required geometric configuration for calculating the electric field due to space charge is illustrated. The rod electrode is modeled as a sphere, centered at the origin of the global reference frame ($\vec{O} = 0\hat{x} + 0\hat{y} + 0\hat{z}$), while the space charge envelope is enclosed within a conical geometry. The apex (tip) of this cone is positioned at $\vec{P}_{apex} = R\hat{z}$ (where R is the radius of the spherical rod tip), and it extends along the central axis of symmetry to a maximum length of $\|\vec{P}_{cs}\| = L_{cs}$ (where cs denotes the **C**orona **S**treamer **B**oundary).

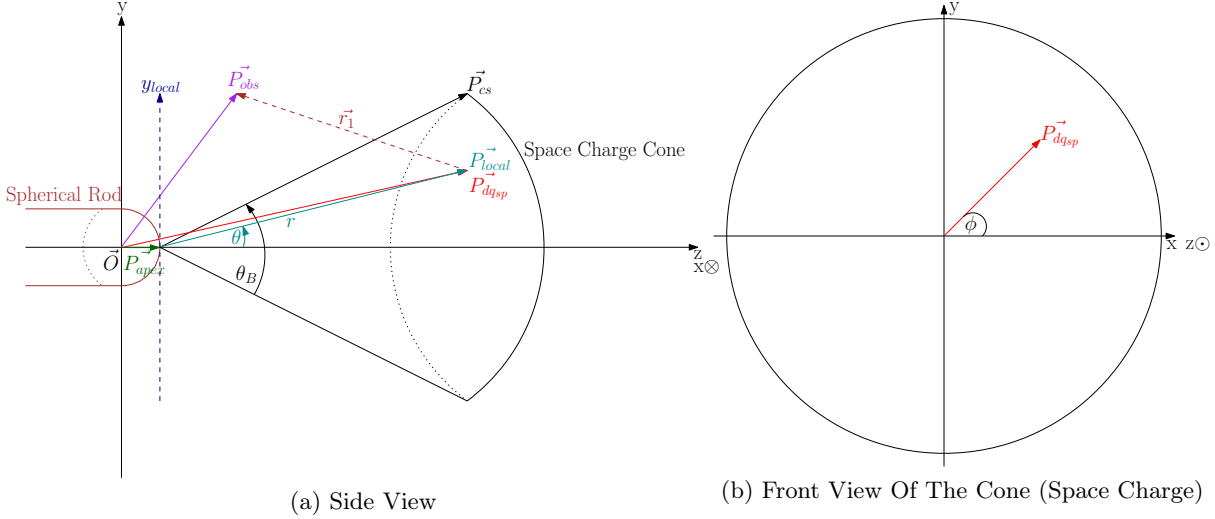


Figure 1: Vector schematic for the spherical rod tip and space charge cone.

2.3.1 Space Charge Electric Field Vector ($\vec{r}_1$)

- **Space Charge Position Vector ($\vec{P}_{dqsp}$):**

Source Characterization: To model the space charge kinematics, an infinitesimally differential charge element $d\mathbf{q}$ within the conical ionization envelope is considered. Assuming a spherical coordinate framework (r, θ, ϕ) anchored at the apex of the discharge cone ($\vec{P}_{apex}$), the position vector $\vec{P}_{local}$ mapping the source charge to the **local reference frame** within the conical envelope (Its center is $\vec{P}_{apex}$), can be calculated as:

$$\vec{P}_{local} = r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y} + r \cos(\theta) \hat{z} \quad (3)$$

The final space charge position vector can be calculated easily (Remember $P_{apex} = R\hat{z}$) as:

$$\vec{P}_{dqsp} = \vec{P}_{apex} + \vec{P}_{local} \Rightarrow \vec{P}_{dqsp} = r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y} + (r \cos(\theta) + R) \hat{z} \quad (4)$$

- **Observation Position Vector ($\vec{P}_{obs}$):**

Field Evaluation Point: To evaluate the resulting electric field intensity and space charge kinematics at any arbitrary location in space, a generic observation point is defined. Expressed natively in the global Cartesian coordinate system, the field observation vector is given by:

$$\vec{P}_{obs} = x\hat{x} + y\hat{y} + z\hat{z} \quad (5)$$

Due to the axial symmetry of the conical envelope, all transverse field components (x and y) cancel out completely upon integration. The analysis can therefore be restricted without loss of generality to an observation point on the central z-axis:

$$\vec{P}_{obs} = z\hat{z} \quad (6)$$

The space charge electric field vector ($\vec{r}_1$) is calculated as:

$$\begin{aligned}\vec{r}_1 &= \vec{P}_{obs} - \vec{P}_{dqsp} \\ \vec{r}_1 &= z\hat{z} - r\sin(\theta)\cos(\phi)\hat{x} - r\sin(\theta)\sin(\phi)\hat{y} - (r\cos(\theta) + R)\hat{z} \\ \vec{r}_1 &= -[r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (r\cos(\theta) + R - z)\hat{z}]\end{aligned}\quad (7)$$

and its magnitude (length):

$$\begin{aligned}r_1 &= \sqrt{r^2\sin^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + [r\cos(\theta) + R - z]^2} \\ r_1 &= \sqrt{r^2\sin^2(\theta)[\cos^2(\phi) + \sin^2(\phi)] + [r\cos(\theta) + R - z]^2} \\ r_1 &= \sqrt{[r\cos(\theta) + R - z]^2 + [r\sin(\theta)]^2}\end{aligned}\quad (8)$$

finally, the unit direction of the field will be:

$$\hat{r}_1 = \frac{\vec{r}_1}{r_1} = -\frac{r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + [r\cos(\theta) + R - z]\hat{z}}{\sqrt{[r\cos(\theta) + R - z]^2 + [r\sin(\theta)]^2}}\quad (9)$$

2.3.2 Final Calculation Of The Space Charge Electric Field ($\vec{E}_{sp}$)

First, we express the infinitesimally differential space charge in terms of the space charge density (2) using the spherical coordinate system:

$$\begin{aligned}dq_{sp} &= \rho(r)dv \\ dq_{sp} &= \rho(r)r^2\sin(\theta)drd\theta d\phi\end{aligned}\quad (10)$$

To ensure a physically consistent integration volume, the boundaries of the spatial variables defining the conical ionization envelope must be explicitly established. The domains are restricted as follows:

- **Radial Distance** ($r \in [0, L_{cs}]$):

Defines the active length of the corona discharge zone, extending from the cone apex ($\vec{P}_{apex}$) to the outer streamer boundary layer (L_{cs}).

- **Polar Angle** ($\theta \in [0, \frac{\theta_B}{2}]$):

Restricted strictly to the semi-vertical angle of the cone apex. Because the volumetric geometry is fully axisymmetric, integrating the polar angle over a half-span prevents redundant geometric overlapping and a double-counting of the differential volume elements (dv).

- **Azimuthal Angle** ($\phi \in [0, 2\pi]$):

Allows a complete angular rotation about the central z-axis, mapping out the entire circular cross-section of the space charge envelope.

$$\text{Domains: } \begin{cases} r \in [0, L_{cs}] \\ \theta \in [0, \theta_B/2] \\ \phi \in [0, 2\pi] \end{cases}\quad (11)$$

then, we apply Coulombs law:

$$\begin{aligned}
d\vec{E}_{sp} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{dq_{sp}}{r_1^2} \hat{r}_1 \\
\vec{E}_{sp} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2 \sin(\theta) d\phi dr d\theta}{r_1^2} \hat{r}_1 \\
\vec{E}_{sp} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2 \sin(\theta)}{r_1^2} \left(-\frac{r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y} + [r \cos(\theta) + R - z] \hat{z}}{r_1} \right) d\phi dr d\theta \\
\vec{E}_{sp} &= -\frac{1}{4\pi\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y} + [r \cos(\theta) + R - z] \hat{z}] d\phi dr d\theta
\end{aligned} \tag{12}$$

Let (breaking down the integration into two terms):

$$\vec{E}_{sp} = -\frac{1}{4\pi\epsilon_r\epsilon_0} [\vec{I}_{sp,1} + \vec{I}_{sp,2}] \tag{13}$$

$$\vec{I}_{sp,1} = \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y}] d\phi dr d\theta \tag{14}$$

$$\vec{I}_{sp,2} = \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \cos(\theta) + R - z] \hat{z} d\phi dr d\theta \tag{15}$$

Showing that $\vec{I}_{sp,1} = 0$:

$$\begin{aligned}
\vec{I}_{sp,1} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y}] d\phi dr d\theta \\
\vec{I}_{sp,1} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} r \sin(\theta) \cos(\phi) d\phi dr d\theta \hat{x} + \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} r \sin(\theta) \sin(\phi) d\phi dr d\theta \hat{y} \\
\vec{I}_{sp,1} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)(r+R)^2}{r_1^3} r \sin(\theta) \sin(\theta) dr d\theta \int_0^{2\pi} \cos(\phi) d\phi \hat{x} + \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)(r+R)^2 \sin(\theta)}{r_1^3} \sin(\theta) dr d\theta \int_0^{2\pi} \sin(\phi) d\phi \hat{y} \\
\vec{I}_{sp,1} &= 0
\end{aligned} \tag{16}$$

Finally, the **Space Charge Electric Field** is given by substituting (16) & (15) to (13):

$$\begin{aligned}
\vec{E}_{sp} &= -\frac{1}{4\pi\epsilon_r\epsilon_0} \left[\vec{I}_{sp,1} + \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \cos(\theta) + R - z] \hat{z} d\phi dr d\theta \right] \\
\vec{E}_{sp} &= -\frac{1}{4\pi\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \cos(\theta) + R - z] \hat{z} d\phi dr d\theta \\
\vec{E}_{sp} &= -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \cos(\theta) + R - z] dr d\theta \hat{z}
\end{aligned} \tag{17}$$

2.3.3 Summarizing

$$\vec{E}_{sp} = -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [r \cos(\theta) + R - z] dr d\theta \hat{z} \quad , \quad r_1 = \sqrt{[r \cos(\theta) + R - z]^2 + [r \sin(\theta)]^2} \tag{18}$$

OR

$$\vec{E}_{sp} = \frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2 \sin(\theta)}{r_1^3} [z - R - r \cos(\theta)] dr d\theta \hat{z} \quad , \quad r_1 = \sqrt{[z - R - r \cos(\theta)]^2 + [r \sin(\theta)]^2} \tag{19}$$

2.4 Calculation Of The Conductor Image Electric Field ($\vec{E}_{I_{rod}}$)

figure 2 illustrates the spatial configuration and coordinate tracking of the induced rod image components. Specifically, it details the position vector of the spherical image charge element ($\vec{P}_{dq'}$) alongside the corresponding field displacement vector ($\vec{r}_4$), which is directed from the source position ($\vec{P}_{dq'}$) towards the observation point ($\vec{P}_{obs}$). The field direction will be determined by the product of $\vec{r}_4$ and the **sign** of the image charge. This section establishes the foundational vector transformation framework required to determine these spatial positions and directional orientations. Using these coordinate mappings, the classical Method of Images for a differential point charge in the presence of a grounded spherical conductor is analytically formulated.

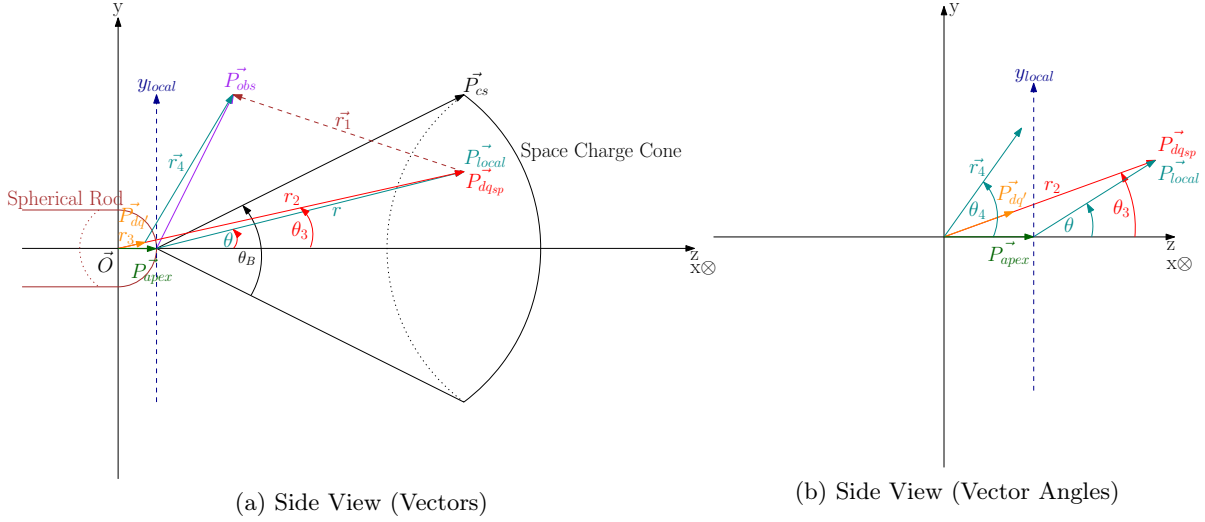


Figure 2: Vector schematic for the conductor (rod) image charge.

2.4.1 Space Charge & The Center Of The Rod-Conductor ($\vec{r}_2$)

In order to construct the position vector of the spherical image charge, we need both the distance & the direction along the line that connects the position of the infinitesimally differential space charge (4) and the center of the rod ($\vec{O} = 0\hat{x} + 0\hat{y} + 0\hat{z}$). This can be achieved as:

$$\begin{aligned}\vec{r}_2 &= \vec{P}_{dqsp} - \vec{O} = \vec{P}_{dqsp} \\ \vec{r}_2 &= r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (r\cos(\theta) + R)\hat{z} - 0 \\ \vec{r}_2 &= r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + [R + r\cos(\theta)]\hat{z}\end{aligned}\quad (20)$$

while its magnitude (length):

$$\begin{aligned}r_2 &= \sqrt{r^2\sin^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + [R + r\cos(\theta)]^2} \\ r_2 &= \sqrt{r^2\sin^2(\theta) \left[\cos^2(\phi) + \sin^2(\phi) \right] + [R + r\cos(\theta)]^2} \\ r_2 &= \sqrt{[R + r\cos(\theta)]^2 + [r\sin(\theta)]^2}\end{aligned}\quad (21)$$

and its unit direction:

$$\hat{r}_2 = \frac{\vec{r}_2}{r_2} = \frac{r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + [R + r\cos(\theta)]\hat{z}}{\sqrt{[R + r\cos(\theta)]^2 + [r\sin(\theta)]^2}}\quad (22)$$

2.4.2 Image Charge Position ($\vec{P}_{dq'}$)

According to the Method of Images for a differential point charge in the presence of a grounded spherical conductor, the image charge (dq') is located at a distance (r_3) away from the center of the conductor:

$$r_3 = \frac{R^2}{r_2} \quad (23)$$

where:

- R : Is the radius of the spherical conductor.
- r_2 : Is the distance between the differential point charge & the center of the spherical conductor.

To maintain absolute theoretical alignment with the analytical framework established by **Wang Wenda & Wang Haiting [1]**, it is necessary to formally define the localized polar angle (θ_3). This angle characterizes the orientation of the alignment vector $\vec{r}_2$ relative to the central vertical z-axis of symmetry. While tracking individual Cartesian coordinates introduces an explicit azimuthal dependency via $\cos(\phi)$ or $\sin(\phi)$, the inherent axisymmetry of the rod-plane gap dictates that the localized polar angle depends globally on the absolute radial displacement away from the vertical axis. By evaluating the norm of the total projection within the horizontal x - y plane, the azimuthal parameter ϕ is analytically eliminated via the fundamental circular identity $\cos^2(\phi) + \sin^2(\phi) = 1$:

$$\begin{aligned} \sin(\theta_3) &= \frac{\|(\vec{r}_2 \cdot \hat{x})\hat{x} + (\vec{r}_2 \cdot \hat{y})\hat{y}\|}{r_2} = \frac{\sqrt{[r\sin(\theta)\cos(\phi)]^2 + [r\sin(\theta)\sin(\phi)]^2}}{r_2} = \frac{\sqrt{r^2\sin^2(\theta) \left[\cos^2(\phi) + \sin^2(\phi) \right]}}{r_2} = \frac{r\sin(\theta)}{r_2} \\ \sin(\theta_3) &= \frac{r\sin(\theta)}{r_2} \Rightarrow r\sin(\theta) = r_2\sin(\theta_3) \\ \theta_3 &= \sin^{-1}\left(\frac{r\sin(\theta)}{r_2}\right) \end{aligned} \quad (24)$$

while:

$$\cos(\theta_3) = \frac{\vec{r}_2 \cdot \hat{z}}{r_2} = \frac{R + r\cos(\theta)}{r_2} \quad (25)$$

The precise position vector that can describe the exact location of the differential image charge can be constructed as:

$$\begin{aligned} \vec{P}_{dq'} &= \vec{O} + r_3\hat{r}_2 = r_3 \frac{r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + [R + r\cos(\theta)]\hat{z}}{r_2} \\ \vec{P}_{dq'} &= \frac{r_3}{r_2} r\sin(\theta)\cos(\phi)\hat{x} + \frac{r_3}{r_2} r\sin(\theta)\sin(\phi)\hat{y} + \frac{r_3}{r_2} [R + r\cos(\theta)]\hat{z} \end{aligned} \quad (26)$$

2.4.3 Image Charge To Observation Displacement Vector ($\vec{r}_4$)

Finally, the Image Charge To Observation Displacement Vector can be calculated as:

$$\begin{aligned}\vec{r}_4 &= \vec{P}_{obs} - \vec{P}_{dq'} \\ \vec{r}_4 &= z\hat{z} - \frac{r_3}{r_2}r\sin(\theta)\cos(\phi)\hat{x} - \frac{r_3}{r_2}r\sin(\theta)\sin(\phi)\hat{y} - \frac{r_3}{r_2}[R + r\cos(\theta)]\hat{z} \\ \vec{r}_4 &= -\frac{r_3}{r_2}r\sin(\theta)\cos(\phi)\hat{x} - \frac{r_3}{r_2}r\sin(\theta)\sin(\phi)\hat{y} - \left\{ \frac{r_3}{r_2}[R + r\cos(\theta)] - z \right\} \hat{z}\end{aligned}\quad (27)$$

and the angle of the orientation of the vector $\vec{r}_4$ relative to the central vertical z-axis of symmetry:

$$\begin{aligned}\sin(\theta_4) &= \frac{\|(\vec{r}_4 \cdot \hat{x})\hat{x} + (\vec{r}_4 \cdot \hat{y})\hat{y}\|}{r_4} = \frac{\sqrt{\left[\frac{r_3}{r_2}r\sin(\theta)\cos(\phi)\right]^2 + \left[\frac{r_3}{r_2}r\sin(\theta)\sin(\phi)\right]^2}}{r_4} \\ \sin(\theta_4) &= \frac{\sqrt{\left(\frac{r_3}{r_2}\right)^2 r^2 \sin^2(\theta) \left[\cos^2(\phi) + \sin^2(\phi)\right]}}{r_4} = \frac{\frac{r_3}{r_2}r\sin(\theta)}{r_4} = \frac{r_3 \cdot \frac{r\sin(\theta)}{r_2}}{r_4} \xrightarrow{\sin(\theta_3)} \frac{r_3 \sin(\theta_3)}{r_4} \\ \theta_4 &= \sin^{-1}\left(\frac{r_3 \sin(\theta_3)}{r_4}\right)\end{aligned}\quad (28)$$

While evaluating the directional orientation relative to the central vertical axis, the absolute value is applied to the numerator to ensure a positive scalar projection along the $\hat{z}$ orientation, yielding:

$$\begin{aligned}\cos(\theta_4) &= \frac{\vec{r}_4 \cdot \hat{z}}{r_4} \\ \cos(\theta_4) &= \frac{\left| \frac{r_3}{r_2}[R + r\cos(\theta)] - z \right|}{r_4} = \frac{\left| z - r_3 \frac{[R + r\cos(\theta)]}{r_2} \right|}{r_4} \xrightarrow{\cos(\theta_3)} \frac{z - r_3 \cos(\theta_3)}{r_4} \\ \cos(\theta_4) &= \frac{z - r_3 \cos(\theta_3)}{r_4}\end{aligned}\quad (29)$$

while its magnitude (length):

$$\begin{aligned}r_4 &= \sqrt{\left[\frac{r_3}{r_2}r\sin(\theta)\cos(\phi)\right]^2 + \left[\frac{r_3}{r_2}r\sin(\theta)\sin(\phi)\right]^2 + \left(\frac{r_3}{r_2}[R + r\cos(\theta)] - z\right)^2} \\ r_4 &= \sqrt{\left(\frac{r_3}{r_2}r\sin(\theta)\right)^2 \left[\cos^2(\phi) + \sin^2(\phi)\right] + \left(\frac{r_3}{r_2}[R + r\cos(\theta)] - z\right)^2} \\ r_4 &= \sqrt{\left(\frac{r_3}{r_2}r\sin(\theta)\right)^2 + \left(\frac{r_3}{r_2}[R + r\cos(\theta)] - z\right)^2}\end{aligned}\quad (30)$$

and since $r\sin(\theta) = r_2\sin(\theta_3)$ (24) & $\frac{R+r\cos(\theta)}{r_2} = \cos(\theta_3)$ (25) substituting them into (30) we get:

$$\begin{aligned}r_4 &= \sqrt{\left(\frac{r_3}{r_2}r_2\sin(\theta_3)\right)^2 + \{r_3\cos(\theta_3) - z\}^2} \\ r_4 &= \sqrt{[z - r_3\cos(\theta_3)]^2 + [r_3\sin(\theta_3)]^2}\end{aligned}\quad (31)$$

and the unit direction:

$$\hat{r}_4 = \frac{\vec{r}_4}{r_4} = \frac{-\frac{r_3}{r_2}r\sin(\theta)\cos(\phi)\hat{x} - \frac{r_3}{r_2}r\sin(\theta)\sin(\phi)\hat{y} - \left\{ \frac{r_3}{r_2}[R + r\cos(\theta)] - z \right\} \hat{z}}{r_4}\quad (32)$$

2.4.4 Final Calculation Of The Conductor Image Electric Field ($\vec{E}_{I_{rod}}$)

According to the Method of Images for a differential point charge in the presence of a grounded spherical conductor:

$$dq' = -\frac{R}{r_2} dq_{sp} \quad (33)$$

substituting $dq_{sp} = \rho(r)r^2 \sin(\theta) dr d\theta d\phi$ (10) into (33):

$$dq' = -\frac{R}{r_2} \rho(r)r^2 \sin(\theta) dr d\theta d\phi \quad (34)$$

applying Coulumb's law using (34), (32)

$$\begin{aligned} d\vec{E}_{I_{rod}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{dq'}{r_4^2} \hat{r}_4 \\ d\vec{E}_{I_{rod}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{-\frac{R}{r_2} \rho(r)r^2 \sin(\theta) dr d\theta d\phi}{r_4^2} \cdot \frac{-\frac{r_3}{r_2} r \sin(\theta) \cos(\phi) \hat{x} - \frac{r_3}{r_2} r \sin(\theta) \sin(\phi) \hat{y} - \left\{ \frac{r_3}{r_2} [R + r \cos(\theta)] - z \right\} \hat{z}}{r_4} \\ d\vec{E}_{I_{rod}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left(\frac{r_3}{r_2} r \sin(\theta) \cos(\phi) \hat{x} + \frac{r_3}{r_2} r \sin(\theta) \sin(\phi) \hat{y} + \left\{ \frac{r_3}{r_2} [R + r \cos(\theta)] - z \right\} \hat{z} \right) d\phi dr d\theta \end{aligned} \quad (35)$$

let:

$$\vec{E}_{I_{rod}} = \vec{I}_{rod,1} + \vec{I}_{rod,2} \quad (36)$$

$$\vec{I}_{rod,1} = \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left(\frac{r_3}{r_2} r \sin(\theta) \cos(\phi) \hat{x} + \frac{r_3}{r_2} r \sin(\theta) \sin(\phi) \hat{y} \right) d\phi dr d\theta = 0 \quad (37)$$

$$\vec{I}_{rod,2} = \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left\{ \frac{r_3}{r_2} [R + r \cos(\theta)] - z \right\} \hat{z} d\phi dr d\theta \quad (38)$$

so:

$$\begin{aligned} \vec{E}_{I_{rod}} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left\{ \frac{r_3}{r_2} [R + r \cos(\theta)] - z \right\} \hat{z} d\phi dr d\theta \\ \vec{E}_{I_{rod}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} d\theta \int_0^{L_{cs}} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left[r_3 \frac{R + r \cos(\theta)}{r_2} - z \right] dr d\theta \hat{z} \\ \vec{E}_{I_{rod}} &= -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\frac{R}{r_2} \rho(r)r^2 \sin(\theta)}{r_4^3} \left[z - r_3 \frac{R + r \cos(\theta)}{r_2} \right] dr d\theta \hat{z} \\ \vec{E}_{I_{rod}} &= -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{R \rho(r)r^2 \sin(\theta)}{r_4^2 r_2} \cdot \frac{z - r_3 \cos(\theta_3)}{r_4} dr d\theta \hat{z} \\ \vec{E}_{I_{rod}} &= -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{R r^2 \rho(r) \sin(\theta)}{r_4^2 r_2} \cos(\theta_4) dr d\theta \hat{z} \\ \vec{E}_{I_{rod}} &= -\frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{R r^2 \rho(r) \sin(\theta) \cos(\theta_4)}{r_4^2 r_2} dr d\theta \hat{z} \end{aligned} \quad (39)$$

OR

$$\vec{E}_{I_{rod}} = \frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2 R \cdot \sin(\theta)}{r_4^3 r_2} \left[r_3 \frac{R + r \cos(\theta)}{r_2} - z \right] dr d\theta \hat{z} \quad (40)$$

2.4.5 Conclusion

Primary Formulation: Pure Coordinate Representation (Computational Core)

For numerical implementation and discrete simulations, expressing the image electric field directly in the global coordinate system avoids intermediate trigonometric transformations and sign ambiguities. The field is evaluated directly via:

$$\begin{cases} r_4 = \sqrt{\left(z - \frac{r_3}{r_2} [R + r \cos(\theta)]\right)^2 + \left(\frac{r_3}{r_2} r \sin(\theta)\right)^2} \\ \vec{E}_{I_{rod}} = -\frac{1}{2\epsilon_r \epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r) r^2 R \sin(\theta)}{r_4^3 r_2} \left[z - \frac{r_3}{r_2} (R + r \cos(\theta))\right] dr d\theta \hat{z} \end{cases} \quad (41)$$

Alternative Formulation: Intermediate Angle Representation (Wang & Wang Framework)

For analytical comparison with classical literature (e.g., Wang & Wang), the geometry can alternatively be parameterized using the polar transformation angles θ_3 and θ_4 :

$$\begin{cases} r_4 = \sqrt{[z - r_3 \cos(\theta_3)]^2 + [r_3 \sin(\theta_3)]^2} \\ \vec{E}_{I_{rod}} = -\frac{1}{2\epsilon_r \epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{R r^2 \rho(r) \sin(\theta) \cos(\theta_4)}{r_4^3 r_2} dr d\theta \hat{z} \end{cases} \quad (42)$$

Note on Implementation: While both formulations are mathematically identical, the pure coordinate representation is preferred in computational algorithms to minimize floating-point operation overhead (eliminating $\cos^{-1}$, $\sin^{-1}$ calls) and preserve strict sign consistency along the $\hat{z}$ axis.

2.5 Calculation Of The Plane Image Electric Field ($\vec{E}_{I_{plane}}$)

figure 3 illustrates the schematic vector layout used to evaluate the electrostatic field contribution originating from the image charge of the grounded boundary plane located at $\vec{P}_{plane} = d\hat{z}$ where d is the gap distance. The differential element (dq_{sp}) within the localized space charge distribution is specified at spatial coordinate $\vec{P}_{dq_{sp}} = r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (r\cos(\theta) + R)\hat{z}$ (4) via spherical representation relative to the conductor tip (apex), where R denotes the hemispherical tip radius and r, θ, ϕ are the internal local cloud coordinates. All remaining spatial distribution parameters retain their previously defined definitions.

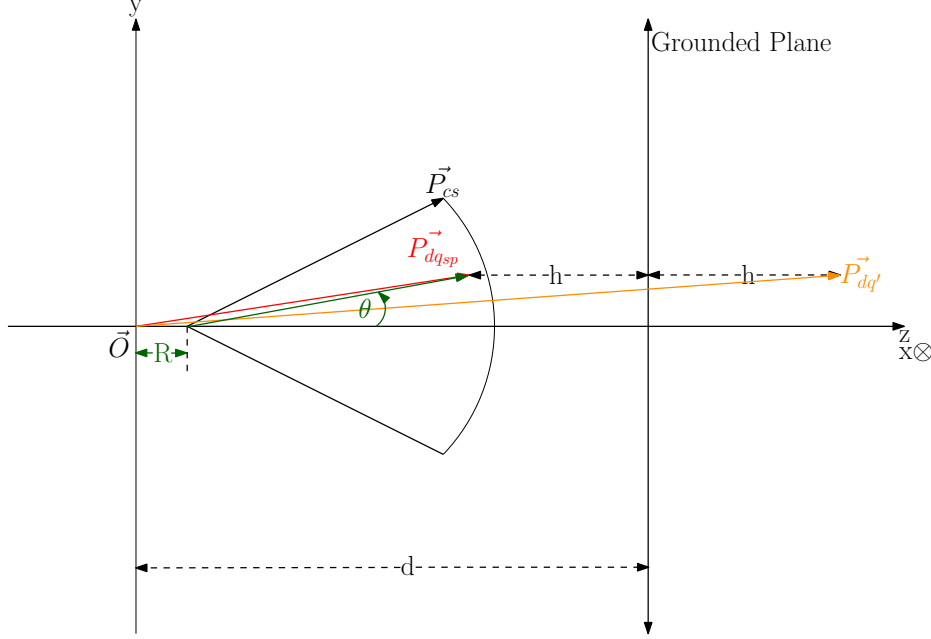


Figure 3: Vector schematic for the plane image charge.

2.5.1 Image Charge Position ($\vec{P}_{dq'}$) & Image Charge (dq')

The grounded conducting plane is oriented normal to the $\hat{z}$ axis at $z = d$. According to the method of image charges, the image distribution reflects solely along the spatial direction normal to the boundary while preserving its transverse x and y coordinates. We define h as the axial separation distance from the discrete space charge element ($\vec{P}_{dq_{sp}}$) to the grounded plane:

$$h = d - \vec{P}_{dq_{sp}} \cdot \hat{z} = d - r\cos(\theta) - R \quad (43)$$

Reflecting the axial position across the boundary plane yields the image charge location:

$$\begin{aligned} \vec{P}_{dq'} &= \vec{P}_{dq_{sp}} + 2h\hat{z} \\ \vec{P}_{dq'} &= r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (r\cos(\theta) + R + 2d - 2r\cos(\theta) - 2R)\hat{z} \\ \vec{P}_{dq'} &= r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R)\hat{z} \end{aligned} \quad (44)$$

while the image charge is just the negated space charge:

$$dq' = -dq_{sp}$$

substituting the space charge distribution (10):

$$dq' = -\rho(r)r^2\sin(\theta)drd\theta d\phi \quad (45)$$

2.5.2 Image Charge To Observation Displacement Vector ($\vec{r}_5$)

The Image Charge To Observation Displacement Vector can be calculated as:

$$\begin{aligned}\vec{r}_5 &= \vec{P}_{obs} - \vec{P}_{dq'} \\ \vec{r}_5 &= z\hat{z} - r\sin(\theta)\cos(\phi)\hat{x} - r\sin(\theta)\sin(\phi)\hat{y} - (2d - r\cos(\theta) - R)\hat{z} \\ \vec{r}_5 &= -[r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R - z)\hat{z}]\end{aligned}\quad (46)$$

while its magnitude (length):

$$\begin{aligned}r_5 &= \sqrt{r^2\sin^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + (2d - r\cos(\theta) - R - z)^2} \\ r_5 &= \sqrt{r^2\sin^2(\theta)[\cos^2(\phi) + \sin^2(\phi)] + (2d - r\cos(\theta) - R - z)^2} \\ r_5 &= \sqrt{[r\sin(\theta)]^2 + (2d - r\cos(\theta) - R - z)^2}\end{aligned}\quad (47)$$

and its unit direction:

$$\begin{aligned}\hat{r}_5 &= \frac{\vec{r}_5}{r_5} \\ \hat{r}_5 &= -\frac{r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R - z)\hat{z}}{\sqrt{[r\sin(\theta)]^2 + (2d - r\cos(\theta) - R - z)^2}}\end{aligned}\quad (48)$$

2.5.3 Final Calculation Of The Plane Image Electric Field ($\vec{E}_{I_{plane}}$)

$$\begin{aligned}d\vec{E}_{I_{plane}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{dq'}{r_5^2} \hat{r}_5 \\ d\vec{E}_{I_{plane}} &= \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{-\rho(r)r^2\sin(\theta)drd\theta d\phi}{r_5^3} \left[-\frac{r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R - z)\hat{z}}{r_5} \right] \\ d\vec{E}_{I_{plane}} &= -\frac{1}{4\pi\epsilon_r\epsilon_0} \frac{-\rho(r)r^2\sin(\theta)}{r_5^3} [r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R - z)\hat{z}] drd\theta d\phi \\ \vec{E}_{I_{plane}} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} [r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y} + (2d - r\cos(\theta) - R - z)\hat{z}] drd\theta d\phi \\ \vec{E}_{I_{plane}} &= \vec{I}_{plane,1} + \vec{I}_{plane,2} \\ \vec{I}_{plane,1} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} [r\sin(\theta)\cos(\phi)\hat{x} + r\sin(\theta)\sin(\phi)\hat{y}] drd\theta d\phi = 0 \\ \vec{I}_{plane,2} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \int_0^{2\pi} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} (2d - r\cos(\theta) - R - z)\hat{z} drd\theta d\phi \\ \vec{I}_{plane,2} &= \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} (2d - r\cos(\theta) - R - z)\hat{z} drd\theta \int_0^{2\pi} d\phi \\ \vec{I}_{plane,2} &= \frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} (2d - r\cos(\theta) - R - z)\hat{z} drd\theta \\ \vec{E}_{I_{plane}} &= \frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2\sin(\theta)}{r_5^3} (2d - r\cos(\theta) - R - z)\hat{z} drd\theta\end{aligned}$$

2.6 Derivation of Space Charge Amplitude A

The axial space charge electric field $\vec{E}_{sp}(z)$ is given by:

$$\vec{E}_{sp}(z) = \frac{1}{2\epsilon_r\epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r)r^2 \sin \theta}{r_1^3} [z - R - r \cos \theta] dr d\theta \hat{z} \quad (49)$$

where $r_1 = \sqrt{[z - R - r \cos \theta]^2 + [r \sin \theta]^2}$ and $\rho(r) = \frac{A}{(r+C)^B}$.

Volumetric Integration for Q_{cs}

Integrating $\rho(r)$ over the conical volume $dV = r^2 \sin \theta dr d\theta d\phi$:

$$Q_{cs} = \int \int \int \rho(r) dv = \int_0^{2\pi} d\phi \int_0^{\frac{\theta_B}{2}} \sin \theta d\theta \int_0^{L_{cs}} \frac{A}{(r+C)^B} r^2 dr \quad (50)$$

Evaluating the angular terms:

$$Q_{cs} = A \cdot 2\pi \left[1 - \cos \left(\frac{\theta_B}{2} \right) \right] \int_0^{L_{cs}} \frac{r^2}{(r+C)^B} dr \quad (51)$$

Gauss's Law at Streamer Boundary

Applying Gauss's Law over a closed surface S enclosing the streamer volume:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{cs}}{\epsilon_0} \quad (52)$$

Assuming the flux is concentrated across the circular head area S_{head} at axial distance L_{cs} where the field magnitude is uniform ($\|\vec{E}\| = E_c$):

$$\iint_{S_{\text{head}}} E_c dA = E_c A_{\text{head}} = \frac{Q_{cs}}{\epsilon_0} \quad (53)$$

From cone trigonometry, the head radius is $r_{\text{head}} = L_{cs} \tan \left(\frac{\theta_B}{2} \right)$, giving the head area:

$$A_{\text{head}} = \pi r_{\text{head}}^2 = \pi \left[L_{cs} \tan \left(\frac{\theta_B}{2} \right) \right]^2 \quad (54)$$

Substituting A_{head} back into Gauss's Law and isolating Q_{cs} :

$$Q_{cs} = \epsilon_0 E_c A_{\text{head}} = \pi \epsilon_0 E_c L_{cs}^2 \tan^2 \left(\frac{\theta_B}{2} \right) \quad (55)$$

Equating and Isolating A

Equating both expressions for Q_{cs} yields the explicit result:

$$A(E_c, B, C, L_{cs}, \theta_B) = \frac{\overbrace{\pi \epsilon_0 E_c L_{cs}^2 \tan^2 \left(\frac{\theta_B}{2} \right)}^{Q_{cs}}}{2\pi \left(1 - \cos \left(\frac{\theta_B}{2} \right) \right) \int_0^{L_{cs}} \frac{r^2}{(r+C)^B} dr} \quad (56)$$

2.7 Summarizing

Space Charge Density Function & Domains:

$$\rho(r) = \frac{A}{(r+C)^B} \left[\frac{C}{m^3} \right], \text{ Domains: } \begin{cases} r \in [0, L_{cs}] \\ \theta \in [0, \theta_B/2] \\ \phi \in [0, 2\pi] \end{cases} \quad (57)$$

$$A(E_c, B, C, L_{cs}, \theta_B) = \frac{\overbrace{\pi \epsilon_0 E_c L_{cs}^2 \tan^2 \left(\frac{\theta_B}{2} \right)}^{Q_{cs}}}{2\pi \left[1 - \cos \left(\frac{\theta_B}{2} \right) \right] \int_0^{L_{cs}} \frac{r^2}{(r+C)^B} dr} \quad (58)$$

B, C are chosen such that $E_{total} = E_{ext} + E_{sp} + E_{I_{rod}} + E_{I_{plane}} = \text{constant (uniform)}$, E_c : Average electric field in the streamer region (typically ~ 5.0 [kV/cm]).

Space Charge ($\vec{E}_{sp}$)

Space charge attenuation distance:

$$r_1 = \sqrt{[z - R - r \cos(\theta)]^2 + [r \sin(\theta)]^2} \quad (59)$$

Space charge electric field:

$$\vec{E}_{sp} = \frac{1}{2\epsilon_r \epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r) r^2 \sin(\theta)}{r_1^3} [z - R - r \cos(\theta)] dr d\theta \hat{z} \quad (60)$$

Rod Image ($\vec{E}_{I_{sp}}$)

Distance between space charge & rod center.

$$r_2 = \sqrt{[R + r \cos(\theta)]^2 + [r \sin(\theta)]^2} \quad (61)$$

Distance of the rod image charge from its center:

$$r_3 = \frac{R^2}{r_2} \quad (62)$$

Rod image charge attenuation distance:

$$r_4 = \sqrt{\left(z - \frac{r_3}{r_2} [R + r \cos(\theta)] \right)^2 + \left(\frac{r_3}{r_2} r \sin(\theta) \right)^2} \quad (63)$$

Rod image electric field:

$$\vec{E}_{I_{rod}} = -\frac{1}{2\epsilon_r \epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r) r^2 R \sin(\theta)}{r_4^3 r_2} \left[z - \frac{r_3}{r_2} (R + r \cos(\theta)) \right] dr d\theta \hat{z} \quad (64)$$

Plane Image ($\vec{E}_{I_{plane}}$)

Plane image charge attenuation distance:

$$r_5 = \sqrt{[r \sin(\theta)]^2 + (2d - r \cos(\theta) - R - z)^2} \quad (65)$$

Plane image electric field:

$$\vec{E}_{I_{plane}} = \frac{1}{2\epsilon_r \epsilon_0} \int_0^{\frac{\theta_B}{2}} \int_0^{L_{cs}} \frac{\rho(r) r^2 \sin(\theta)}{r_5^3} (2d - r \cos(\theta) - R - z) dr d\theta \hat{z} \quad (66)$$

3 Mathematical Equivalence and Reduction to the Wang–Haiting Formulation

To validate the analytical formulation that was developed, it is instructive to demonstrate its exact mathematical equivalence to the foundational space-charge model published by Wang and Haiting [1] (1988). While the Wang–Haiting equations rely on intermediate angular definitions (θ_3, θ_4) and local trigonometric projections, our implementation utilizes coordinate-based algebraic forms designed for computational integration efficiency. This section provides a complete step-by-step proof mapping the field expressions derived in this work to the classic Wang–Haiting equations, accounting for coordinate origin shifts, corrected typographical errors in the original literature, and physical approximations.

3.1 Coordinate System Alignment & Translation

A primary distinction between the two models lies in the choice of global coordinate origin:

1. **Present Model Origin ($\vec{O}$):** Placed at the **center of the hemispherical rod** while the apex of the cone sits at $(0, 0, R)$.
2. **Wang–Haiting Origin ($\vec{O}_{\text{WH}}$):** Placed directly at the **apex of the cone** (rod tip).

Let Z_{WH} represent the axial coordinate measured along the symmetry axis in the Wang–Haiting system. The spatial transformation between the observation coordinate z in our coordinate frame and Z_{WH} is given by:

$$Z_{\text{WH}} = z - R \implies z = Z_{\text{WH}} + R \quad (67)$$

3.2 Reduction of the Direct Space-Charge Field (E_{sp})

From (60), the direct space-charge field component along the axis of symmetry $\hat{z}$ is expressed as:

$$\vec{E}_{sp}(0, 0, z) = \frac{1}{2\epsilon_r \epsilon_0} \int_0^{\theta_B/2} \int_0^{L_{cs}} \frac{\rho(r) r^2 \sin(\theta)}{r_1^3} (z - R - r \cos(\theta)) dr d\theta \hat{z} \quad (68)$$

where r_1 is the algebraic scalar distance from a charge element $\vec{P}_{dq_{sp}}$ to the observation point $(0, 0, z)$:

$$r_1 = \sqrt{[z - R - r \cos(\theta)]^2 + [r \sin(\theta)]^2} \quad (69)$$

Step 1: Origin Translation

Substitute the transformation $z = Z_{\text{WH}} + R$:

$$r_1 = \sqrt{[Z_{\text{WH}} - r \cos(\theta)]^2 + [r \sin(\theta)]^2} = \sqrt{[r \cos(\theta) - Z_{\text{WH}}]^2 + [r \sin(\theta)]^2} \quad (70)$$

and for the field equation:

$$\vec{E}_{sp} = -\frac{1}{2\epsilon_r \epsilon_0} \int_0^{\theta_B/2} \int_0^{L_{cs}} \frac{\rho(r) r^2 \sin(\theta)}{r_1^3} (r \cos(\theta) - Z_{\text{WH}}) dr d\theta \hat{z} \quad (71)$$

Step 2: Verification with Wang–Haiting Equation

In Wang and Haiting (1988), for the space-charge field is given as:

$$E_{sp} = -\frac{1}{2\epsilon} \int_0^{\theta_B/2} \int_0^{L_{cs}} \frac{\rho(r) r^2 \sin(\theta)}{r_1^3} (r \cos(\theta) - Z) dr d\theta \quad (72)$$

with distance kernel:

$$r_1 = \sqrt{(r \cos(\theta) - Z)^2 + [r_3 \sin(\theta)]^2} \quad (73)$$

Note on Literature Typo: The original manuscript published in 1988 contains a typographical error in the distance kernel expression, printing $r_3 \sin \theta$ instead of $r \sin \theta$ in the second term of r_1 . The derivation above confirms that $r \sin \theta$ is the physically correct radial component. Thus, its **identically equivalent** to Wang–Haiting Equation under the coordinate shift $Z = Z_{\text{WH}} = z - R$.

3.3 Reduction of the Rod Image Field ($E_{I_{\text{rod}}}$)

From equation (39), the axial electric field component produced by the hemispherical rod image charge distribution is derived as:

$$\vec{E}_{I_{\text{rod}}}(0, 0, z) = -\frac{1}{2\epsilon_r \epsilon_0} \int_0^{\theta_B/2} \int_0^{L_{cs}} \frac{R r^2 \rho(r) \sin(\theta) \cos(\theta_4)}{r_4^2 r_2} dr d\theta \hat{z} \quad (74)$$

where $r_2 = \sqrt{(R + r \cos \theta)^2 + (r \sin \theta)^2}$ represents the radial distance from the coordinate origin $\vec{O}$ to the source element. The image charge position angle θ_3 , observer distance r_4 , and directional projection angle θ_4 are defined relative to the symmetry axis as:

$$\cos(\theta_3) = \frac{R + r \cos(\theta)}{r_2}, \quad \sin(\theta_3) = \frac{r \sin(\theta)}{r_2} \quad \cos(\theta_4) = \frac{|z - r_3 \cos(\theta_3)|}{r_4} \quad (75)$$

$$r_4 = \sqrt{[z - r_3 \cos(\theta_3)]^2 + [r_3 \sin(\theta_3)]^2} \quad (76)$$

with $r_3 = R^2/r_2$ defining the radial offset of the Kelvin image point.

Step 1: Origin Shift to the Wang–Haiting Coordinate Frame

Applying the coordinate translation from our origin at the sphere center to the Wang–Haiting apex origin ($Z_{\text{WH}} = z - R \Rightarrow z = Z_{\text{WH}} + R$):

$$z - r_3 \cos(\theta_3) = Z_{\text{WH}} + R - r_3 \cos(\theta_3) \quad (77)$$

Substituting into the expression for $\cos \theta_4$ yields:

$$\cos(\theta_4) = \frac{Z_{\text{WH}} + R - r_3 \cos(\theta_3)}{r_4} \quad (78)$$

Step 2: Direct Reduction to Wang–Haiting Formulation

The field equations match directly (using $\epsilon = \epsilon_r \epsilon_0$):

$$E_I = -\frac{1}{2\epsilon} \int_0^{\theta_B/2} \int_0^{L_{cs}} \frac{R r^2 \rho(r) \sin(\theta) \cos(\theta_4)}{r_4^2 r_2} dr d\theta \quad (79)$$

where the image distance kernel r_4 maps identically to:

$$r_4 = \sqrt{(Z_{\text{WH}} + R - r_3 \cos(\theta_3))^2 + (r_3 \sin(\theta_3))^2} \quad (80)$$

Note on Literature Typo: The original Wang and Haiting (1988) paper misprinted the second term inside the r_4 square root as $(r_3 \sin(\theta))^2$ instead of $(r_3 \sin(\theta_3))^2$. The derivation above confirms that θ_3 is the correct angular argument.

This completes the proof that our coordinate-based implementation reduces directly to the classic Wang–Haiting formulation without requiring explicit evaluation of intermediate polar angle transformations during integration.

3.4 Grounded Plane Image Field ($E_{I_{\text{plane}}}$) and Boundary Approximations

In long rod-plane gap discharge modeling (where gap distance $d \gg R$), the high-voltage ionization phenomena and streamer propagation characteristics occur primarily in the immediate vicinity of the rod tip. In the formulation of Wang and Haiting (1988), the contribution of the grounded plane image field $\vec{E}_{I_{\text{plane}}}$ is omitted ($\vec{E}_{I_{\text{plane}}} \approx 0$). Under this assumption, the total space-charge field along the gap axis reduces to:

$$\vec{E}_{\text{total, WH}} = \vec{E}_{ex} + \vec{E}_{sp} + \vec{E}_{I_{\text{rod}}} \quad (81)$$

In our complete model, the plane image field $\vec{E}_{I_{\text{plane}}}$ is explicitly retained to preserve exact boundary conditions across the entire inter-electrode gap length d . When evaluating field distributions near the rod electrode ($z \ll d$), $\vec{E}_{I_{\text{plane}}} \rightarrow 0$.

3.5 Summary of Analytical Equivalence

Field Component	Present Optimization (Sec. 2.6)	Classic Wang-Haiting (1988)	Status
Origin Point	Center of Spherical Tip $\vec{O} = (0, 0, 0)$	Apex of Spherical Tip $\vec{O}_{\text{WH}} = (0, 0, R)$	Shift $z = Z_{\text{WH}} + R$
Direct Field ($\vec{E}_{sp}$)	Vector distance r_1 in Cartesian components	Integral over $[r \cos(\theta) - Z] / r_1^3$	Proven Equivalent
Rod Image ($\vec{E}_{I_{\text{rod}}}$)	Algebraic scalar product $(z - z_{\text{image}}) / r_4^3$	Trigonometric projection $\cos \theta_4 / r_4^2$	Proven Equivalent
Plane Image ($\vec{E}_{I_{\text{plane}}}$)	Explicitly integrated over gap boundary d	Neglected ($\vec{E}_{I_{\text{plane}}} \approx 0$) near rod tip	Exact Generalization

Table 1: Comparison and reduction summary between computational equations and Wang-Haiting.

4 Dynamic Kinematics and Relative Coordinate Tracking

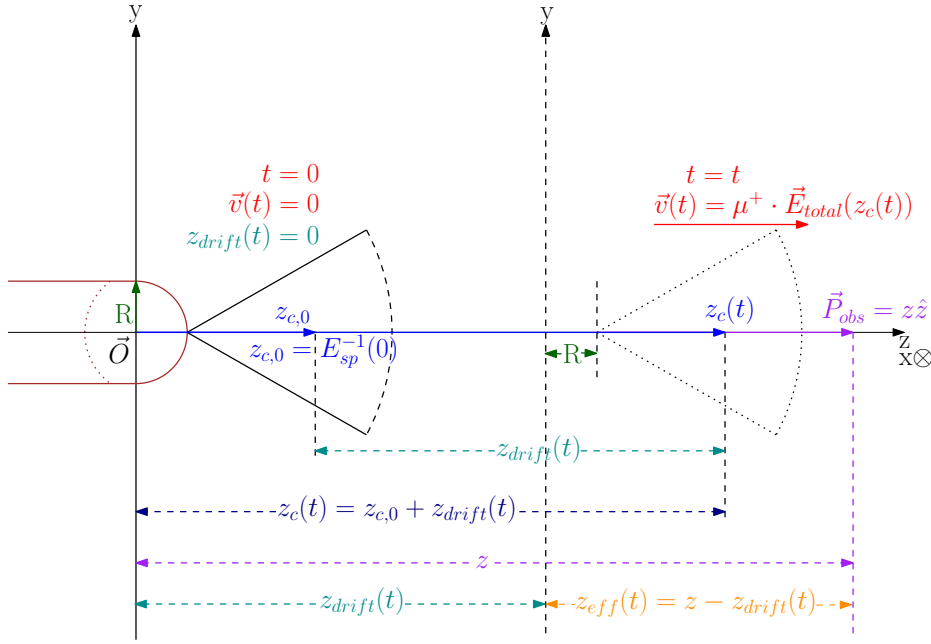


Figure 4: Space Charge Kinematics

To incorporate the spatial translation of the drifting space charge cloud along the primary axis of symmetry, the kinematics are integrated into the existing framework using a **continuous relative coordinate mapping**. Let $z_{drift}(t)$ represent the time-dependent, absolute physical translation of the space charge cloud's center of mass along the positive $\hat{z}$ direction away from its initial coordinate. The actual position of any discrete charge element shifts dynamically over time according to:

$$z_c(t) = z_{c,0} + z_{drift}(t) \quad , \quad E_{sp}(z_{c,0}) = 0 \implies z_{c,0} = E_{sp}^{-1}(0) \quad (82)$$

Because the governing electrostatic field equations depend strictly on the relative displacement vector directed from the source charge coordinate to the target evaluation point ($\vec{P}_{observer} - \vec{P}_{charge,source}$), the physical vertical tracking component transforms as follows:

$$\begin{aligned} \text{Axial Displacement} &= z - z_c(t) \\ &= z - (z_{c,0} + z_{drift}(t)) \\ &= (z - z_{drift}(t)) - z_{c,0} \end{aligned} \quad (83)$$

This algebraic grouping demonstrates that physically moving the space charge cloud forward by $+z_{drift}(t)$ is mathematically identical to keeping the cloud's integration grid completely stationary and shifting the effective observation coordinate backward by $-z_{drift}(t)$.

Consequently, the simulation engine preserves the static computational geometry of the mesh grid. To evaluate the resulting fields at an absolute location z , the framework simply passes a time-dependent effective observer coordinate into the static field integral algorithms:

$$z_{eff}(t) = z - z_{drift}(t) \quad (84)$$

This spatial substitution variable replaces the literal observer coordinate z across all localized field equations, eliminating the computational overhead of dynamically rebuilding or translating multi-dimensional arrays during runtime.

4.1 Kinematic Time-Stepping and Discrete Integration

The dynamic translation $z_{\text{drift}}(t)$ of the space charge cloud is governed by the time-varying total driving electric field evaluated at the moving center of mass $z_c(t)$. By framing the translation explicitly as a continuous function of time, the total accumulated drift distance at any instant t is resolved directly by integrating the cloud's velocity:

$$z_{\text{drift}}(t) = \int_0^t \underbrace{\mu^+ [E_{\text{ex}}(z_c(\tau)) + E_{\text{I,rod}}(z_c(\tau)) + E_{\text{I,plane}}(z_c(\tau))]}_{\|\vec{v}\| \text{ [m/s]: The speed of the cloud's center of mass}} d\tau \quad [\text{m}] \quad (85)$$

where the baseline mobility coefficient for positive ions in atmospheric air is defined as:

$$\mu^+ = 1.4 \times 10^{-4} \left[\frac{\text{m}^2}{\text{V} \cdot \text{s}} \right] \quad (86)$$

Decoupled Coordinate Mapping and Kinematic Reference Tracking

To progress the system chronologically from time t to $t + \Delta t$, the continuous integration limits are partitioned according to the additive property of definite integrals:

$$z_{\text{drift}}(t + \Delta t) = z_{\text{drift}}(t) + \int_t^{t+\Delta t} \mu^+ [E_{\text{ex}}(z_c(\tau)) + E_{\text{I,rod}}(z_c(\tau)) + E_{\text{I,plane}}(z_c(\tau))] d\tau \quad (87)$$

The second term on the right-hand side represents the exact incremental spatial displacement $z_m(t)$ [1] over the step interval Δt . Defining this step displacement yields:

$$z_m(t) = \int_t^{t+\Delta t} \mu^+ [E_{\text{ex}}(z_c(\tau)) + E_{\text{I,rod}}(z_c(\tau)) + E_{\text{I,plane}}(z_c(\tau))] d\tau \quad (88)$$

Substituting this definition back into the partitioned integral yields the clean time-dependent kinematic evolution law:

$$z_{\text{drift}}(t + \Delta t) = z_{\text{drift}}(t) + z_m(t) \quad (89)$$

This mathematical formulation bridges continuous classical kinematics with discrete computational execution. The simulation engine simply advances a master time variable $t \rightarrow t + \Delta t$, which inherently maps the updated spatial boundaries for $z_{\text{drift}}(t)$, $z_c(t)$, and $z_{\text{eff}}(t)$ without relying on software-specific accumulator variables.

5 Sources

[1] Wang Wenduan, Wang Haiting “A New Model For Simulating The Effects Of Space Charge In Gas Discharge”.