A TOY SCALAR FIELD THEORY OF THREE INTERACTING PARTICLES

Abstract

We construct a scalar field theory with three fictitious particles all of equal mass and observe what structure of interactions arise. We find that not all of the interactions occur independently and that identical particles take part in more than one interactions. We also find that the number of interactions that are predicted by intuition alone turn out to be more than what is actually computed. We set up the theory using the LSZ reduction formula and utilize the Kelin Gordan equation as well as the Lagrangian for a free scalar field to analyze our imagined scenario.

1. Initial Setup

We begin with a scalar field theory whose equations of motion are determined by the Klein Gordon equation. The lagrangian is then:

$$L = -\frac{1}{2}\partial^{\mu}\phi\partial_{\mu} - \frac{1}{2}m^2\phi^2 + \Omega_0 \tag{1.1}$$

Varying the action with respect to this Lagrangian gives us our equations of motion:

$$0 = \delta_s = \int d^4x[-\frac{1}{2}\partial^{\mu}\delta_{\phi}\partial_{\mu}\phi - \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\delta_{\phi} - m^2\phi\delta_{\phi}] = \int d^4x[\partial^{\mu}\partial_{\mu}\phi - m^2\phi]\delta_{\phi} \tag{1.2}$$

Which yields the following equation of motion:

$$(-\partial^2 + m^2)\phi(x) = 0 \tag{1.3}$$

To construct our theory, we implement the use of the LSZ reduction formula which allows us to work in momentum space. In the initial construction of the equation we name our three fictitious particles alpha, beta and gamma all with the same properties.

The LSZ Reduction formula then reads:

$$< f|i >= i^{\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'} \int d^4x_{\alpha}e^{ik_{\alpha}x_{\alpha}}(-\partial_{\alpha}^2 + m^2)...d^4x'_{\alpha}e^{ik'_{\alpha}x'_{\alpha}}(-\partial_{\alpha}^{2\prime} + m^2)...$$

$$d^4x_{\beta}e^{ik_{\beta}x_{\beta}}(-\partial_{\beta}^2 + m^2)...d^4x'_{\beta}e^{ik'_{\beta}x'_{\beta}}(-\partial_{\beta}^{2\prime} + m^2)...d^4x_{\gamma}e^{ik_{\gamma}x_{\gamma}}(-\partial_{\gamma}^2 + m^2)...$$

$$.d^4x'_{\gamma}e^{ik'_{\gamma}x'_{\gamma}}(-\partial_{\gamma}^{2\prime} + m^2)...X < 0|T, \phi(x_{\alpha}), \phi(x_{\beta}), \phi(x_{\gamma})|0 > \tag{1.4}$$

There is one important thing to note: The use of this equation requires a slight modification to the Lagrangian. Namely that of reshifting and rescaling. This is because the following two conditions must be satisfied for the LSZ formula to be used:

$$< 0|\phi(x)|0 > \tag{1.5}$$

$$< k|\phi(x)|0 >= e^{-ikx} \tag{1.6}$$

A modified Lagrangian that satisfies these conditions could take a form like this:

$$L = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 + Y\phi \tag{1.7}$$

We do not compute the exact constant coefficients here but it is an exercise that can be done to get the exact Lagrangian. It should be stressed that, despite the Lagrangian being modified in this way, this has no effect on the equation of motion that is derived, namely the Klein Gordon equation.

Now that the initial set up is complete, we begin our computation. The left hand side of the formula is set up in a completely arbitrary fashion. This is because we can invoke the free propagators needed in any way we wish. We can write down any particle-particle interaction we want because no constraints are imposed by the left hand term. Therefore, it is imperative that we begin our analysis with the right hand time ordered term. The way we write out and compute the time ordered term will determine the interactions and the nature of those interactions. Thus we begin our analysis with a computation of the timer ordered term and see what interactions arise. Then, we can fill in the blanks and create the free propagators that are necessary to make sure the same interactions are carried out.

2. The Time Ordered Term

Filling in our Quantum Field functions, we can write out the time ordered term as follows:

$$< 0|T, \int \frac{d^3k_\alpha}{(2\pi)^3 2\omega} \; [a(k_\alpha)e^{ik_\alpha x_\alpha} + a^\dagger(k'_\alpha)e^{-ik_\alpha' x'_\alpha}] \int \frac{d^3k_\beta}{(2\pi)^3 2\omega}$$
$$[a(k_\beta)e^{ik_\beta x_\beta} + a^\dagger(k'_\beta)e^{-ik_\beta' x'_\beta}] \int \frac{d^3k_\gamma}{(2\pi)^3 2\omega} \; [a(k_\gamma)e^{ik_\gamma x_\gamma} + a^\dagger(k'_\gamma)e^{-ik_\gamma' x'_\gamma}]|0> \tag{2.1}$$

Now we clean up this expression by condensing the normalization constants:

$$\frac{dk}{(2\pi)^3 2\omega} \rightarrow \mathbf{dk} \tag{2.2}$$

Now the expression becomes:

$$< 0|T, \int\int\int \mathbf{dk}_\alpha \mathbf{dk}_\beta \mathbf{dk}_\gamma [a(k_\alpha)e^{ik_\alpha x_\alpha} + a^\dagger(k'_\alpha)e^{-ik_\alpha' x'_\alpha}]$$
$$[a(k_\beta)e^{ik_\beta x_\beta} + a^\dagger(k'_\beta)e^{-ik_\beta' x'_\beta}][a(k_\gamma)e^{ik_\gamma x_\gamma} + a^\dagger(k'_\gamma)e^{-ik_\gamma' x'_\gamma}]|0> \tag{2.3}$$

In order to obtain a picture of the interactions, we must expand out the creation and annihilation operators with the hope of obtaining commutation relations. It becomes clear here however that searching for commutation relations will not be the right strategy because expanding out the operators only yields positive signs. Therefore, we will expand out the operators with the intentions of constructing anticommutators and obtaining the interaction picture.

Expanding out the operators:

$$[a(k_\alpha)e^{ik_\alpha x_\alpha}+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}][a(k_\beta)e^{ik_\beta x_\beta}+a^\dagger(k'_\beta)e^{-ik_\beta{}'x'_\beta}][a(k_\gamma)e^{ik_\gamma x_\gamma}+a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}]=$$

$$a(k_\alpha)e^{ik_\alpha x_\alpha}a(k_\beta)e^{ik_\beta x_\beta}+a(k_\alpha)e^{ik_\alpha x_\alpha}a^\dagger(k'_\beta)e^{-ik_\beta{}'x'_\beta}+a(k_\gamma)e^{ik_\gamma x_\gamma}a(k_\alpha)e^{ik_\alpha x_\alpha}+a(k_\alpha)e^{ik_\alpha x_\alpha}a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}$$

$$+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}a(k_\beta)e^{ik_\beta x_\beta}+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}a^\dagger(k'_\beta)e^{-ik_\beta{}'x'_\beta}+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}a(k_\gamma)e^{ik_\gamma x_\gamma}+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}$$

$$+a(k_\beta)e^{ik_\beta x_\beta}a(k_\alpha)e^{ik_\alpha x_\alpha}+a(k_\beta)e^{ik_\beta x_\beta}a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}+a(k_\beta)e^{ik_\beta x_\beta}a(k_\gamma)e^{ik_\gamma x_\gamma}+a(k_\beta)e^{ik_\beta x_\beta}a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}$$

$$+a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}a(k_\alpha)e^{ik_\alpha x_\alpha}+a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}+a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}a(k_\gamma)e^{ik_\gamma x_\gamma}+a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}$$

$$+a(k_\gamma)e^{ik_\gamma x_\gamma}a(k_\alpha)e^{ik_\alpha x_\alpha}+a(k_\gamma)e^{ik_\gamma x_\gamma}a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}+a(k_\gamma)e^{ik_\gamma x_\gamma}a(k_\beta)e^{ik_\beta x_\beta}+a(k_\gamma)e^{ik_\gamma x_\gamma}a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}$$

$$+a^\dagger(k'_\gamma)e^{-ik_\gamma x'_\gamma}a(k_\alpha)e^{ik_\alpha x_\alpha}+a^\dagger(k'_\gamma)e^{-ik_\gamma x'_\gamma}a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}+a^\dagger(k'_\gamma)e^{-ik_\gamma x'_\gamma}a(k_\beta)e^{ik_\beta x_\beta}+a^\dagger(k'_\gamma)e^{-ik_\gamma x'_\gamma}a^\dagger(k'_\beta)e^{-ik_\beta x'_\beta}$$

(2.4)

Now we regroup the terms in this expansion to give us the anti commutation relations we desire:

$$[a(k_\alpha)e^{ik_\alpha x_\alpha}+a^\dagger(k'_\alpha)e^{-ik_\alpha{}'x'_\alpha}][a(k_\beta)e^{ik_\beta x_\beta}+a^\dagger(k'_\beta)e^{-ik_\beta{}'x'_\beta}][a(k_\gamma)e^{ik_\gamma x_\gamma}+a^\dagger(k'_\gamma)e^{-ik_\gamma{}'x'_\gamma}]=$$

$$e^{ik_\alpha x_\alpha}e^{ik_\beta x_\beta}+e^{ik_\beta x_\beta}e^{ik_\alpha x_\alpha}+\{a(k_\alpha),a(k_\beta)\}+e^{ik_\alpha x_\alpha}e^{-ik_\beta{}'x'_\beta}+e^{-ik_\beta{}'x'_\beta}e^{ik_\alpha x_\alpha}\left\{a(k_\alpha),a^\dagger(k'_\beta)\right\}$$

$$+e^{ik_\alpha x_\alpha}e^{-ik_\gamma{}'x'_\gamma}+e^{-ik_\gamma{}'x'_\gamma}e^{ik_\alpha x_\alpha}+\left\{a(k_\alpha),a^\dagger(k'_\gamma)\right\}+e^{ik_\alpha x_\alpha}e^{ik_\gamma x_\gamma}+e^{ik_\gamma x_\gamma}e^{ik_\alpha x_\alpha}+\{a(k_\alpha),a(k_\gamma)\}$$

$$+e^{-ik_\alpha{}'x'_\alpha}e^{ik_\beta x_\beta}+e^{ik_\beta x_\beta}e^{-ik_\alpha{}'x'_\alpha}+\left\{a^\dagger(k'_\alpha),a(k_\beta)\right\}+e^{-ik_\alpha{}'x'_\alpha}e^{-ik_\beta{}'x'_\beta}+e^{-ik_\beta{}'x'_\beta}e^{-ik_\alpha{}'x'_\alpha}+\left\{a^\dagger(k'_\alpha),a^\dagger(k'_\beta)\right\}$$

$$+e^{-ik_\alpha{}'x'_\alpha}e^{ik_\gamma x_\gamma}+e^{ik_\gamma x_\gamma}e^{-ik_\alpha{}'x'_\alpha}+\left\{a^\dagger(k'_\alpha),a(k_\gamma)\right\}+e^{-ik_\alpha{}'x'_\alpha}e^{-ik_\gamma{}'x'_\gamma}+e^{-ik_\gamma{}'x'_\gamma}e^{-ik_\alpha{}'x'_\alpha}+\left\{a^\dagger(k'_\alpha),a^\dagger(k'_\gamma)\right\}$$

$$+e^{ik_\beta x_\beta}e^{ik_\gamma x_\gamma}+e^{ik_\gamma x_\gamma}e^{ik_\beta x_\beta}+\{a(k_\beta),a(k_\gamma)\}+e^{ik_\beta x_\beta}e^{-ik_\gamma{}'x'_\gamma}+e^{-ik_\gamma{}'x'_\gamma}e^{ik_\beta x_\beta}+\left\{a(k_\beta),a^\dagger(k'_\gamma)\right\}$$

$$+e^{-ik_\beta{}'x'_\beta}e^{ik_\gamma x_\gamma}+e^{ik_\gamma x_\gamma}e^{-ik_\beta{}'x'_\beta}+\left\{a^\dagger(k'_\beta),a(k_\gamma)\right\}+e^{-ik_\beta{}'x'_\beta}e^{-ik_\gamma{}'x'_\gamma}+e^{-ik_\gamma{}'x'_\gamma}e^{-ik_\beta{}'x'_\beta}+\left\{a^\dagger(k'_\beta),a^\dagger(k'_\gamma)\right\}$$

(2.5)

Of course, the next natural step would be to evaluate the anti commutators and that is a necessary task. However, it is important at this stage to make some remarks in terms of what we should look for to ensure consistency. When constructing the right hand term of our expression, we can forecast how many free propagators we will have. this , at the very least, gives us a reference for the number of interactions we should expect and should be left with in the time ordered term. Three fictitious particles means that there are nine possible combinations of interactions that can occur. However, this does not take into account self interacting propagators which vanish. There are three of these. You would think that we should expect a total of 6 different interactions after the commutators are evaluated in the time ordered term. However, the computation presents something quite interesting. As long as the number of interactions matches on both sides of the expression the physical insight gained is correct. These interactions will present themselves as Dirac Delta functions, which should also match up when we evaluate the propagators. We use the following computations for anticommutators:

$$\{a(k), a(k')\} = 0 \tag{2.6.1}$$

$$\left\{a^\dagger(k), a^\dagger(k')\right\} = 0 \tag{2.6.2}$$

$$\left\{a(k), a^\dagger(k')\right\} = (2\pi^3 2\omega)\delta^{(3)}(p - p') \tag{2.6.3}$$

Now we evaluate the commutators and ignore the exponential terms as they are not important in extracting the physics that is happening. We also ignore the normalization coefficients that appear in the delta functions themselves:

$$\delta^{(3)}(p_\alpha - p'_\beta) + \delta^{(3)}(p_\alpha - p'_\gamma) + \delta^{(3)}(p'_\alpha - p_\beta) + \delta^{(3)}(p_\beta - p'_\gamma) + \delta^{(3)}(p'_\beta - p_\gamma) \tag{2.7}$$

Summing up the delta functions cancels out the P beta term and we are left with the following outline of interactions:

$$\delta^{(3)}(p_\alpha - p'_\beta) + \delta^{(3)}(p_\alpha - p'_\gamma) + \delta^{(3)}(p'_\alpha - p'_\gamma) + \delta^{(3)}(p'_\beta - p_\gamma) \tag{2.8}$$

Superficially we have obtained some logistics regarding our interactions. We now know that there are 4 interactions that occur. We also know which particles in specific are acting with which. Of course, there is a lot more that can be said about the physics that has been extracted from the time ordered term. But it is best saved for a discussion at the end, after the free propagators have been evaluated and all of the mathematics checks out, thus confirming that the theory has been constructed correctly.

3. Evaluating the Free Propagators

We know turn our attention to the left hand term and begin to evaluate our propagators. It is important to note that the propagators are indeed set up completely artificially. This is why we began with the time ordered term. Once we set up the propagators we will rely on that mathematics working itself out so that the interactions are indeed valid and physical. So it begs the question: what mathematics are we expecting to appear once this term is evaluated? We should expect to recover 4 identical dirac delta functions just as we did when the time ordered term was computed. It is also important to note that the free propagator comes in the form of a Green's function i.e. D (x-y) in position space. We also impose the fact that one of the 5 delta functions in our analysis canceled out, leaving us with 4 integrals to compute.

Writing out the right hand term in terms of free propagators:

$$\int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2 + m^2)D(x_\alpha - x'_\beta) \int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2 + m^2)D(x_\alpha - x'_\gamma)$$
$$\int d^4x'_\alpha e^{ik'_\alpha x'_\alpha}(-\partial_\alpha^{2\prime} + m^2)D(x'_\alpha - x'_\gamma{}') \int d^4x'_\beta e^{ik'_\beta x'_\beta}(-\partial_\beta^{2\prime} + m^2)D(x'_\beta - x_\gamma) \tag{3.1}$$

Now we add in our source terms i.e. J(x) as an intermediate step before computing the integrals:

$$\int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2+m^2)J(x_\alpha)D(x_\alpha-x'_\beta)J(x'_\beta)\int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2+m^2)J(x_\alpha)D(x_\alpha-x'_\beta)J(x'_\beta)$$

$$\int d^4x'_\alpha e^{ik'_\alpha x'_\alpha}(-\partial_\alpha^{2\prime}+m^2)J(x'_\alpha)D(x'_\alpha-x'_\gamma{}')J(x'_\gamma)\int d^4x'_\beta e^{ik'_\beta x'_\beta}(-\partial_\beta^{2\prime}+m^2)J(x'_\beta)D(x'_\beta-x_\gamma)J(x_\gamma)$$

(3.2)

Now we can rewrite our propagators in the form of simple Gaussian integrals*:

$$\int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2+m^2)\int \mathbf{dk}_\alpha e^{ik_\alpha(x_\alpha-x'_\beta)}\int d^4x_\alpha e^{ik_\alpha x_\alpha}(-\partial_\alpha^2+m^2)\int \mathbf{dk}_\alpha e^{ik_\alpha(x_\alpha-x'_\gamma)}$$

$$\int d^4x'_\alpha e^{ik'_\alpha x'_\alpha}(-\partial_\alpha^{2\prime}+m^2)\int \mathbf{dk}'_\alpha e^{ik'_\alpha(x'_\alpha-x'_\gamma)}\int d^4x'_\beta e^{ik'_\beta x'_\beta}(-\partial_\beta^{2\prime}+m^2)\int \mathbf{dk}'_\beta e^{ik'_\beta(x'_\beta-x_\gamma)}$$

(3.3)

Now we can evaluate the gaussian integrals:

$$\int\int\int\int d^4x_\alpha d^4x_\alpha d^4x'_\alpha d^4x'_\beta e^{ik_\alpha x_\alpha}e^{ik_\alpha x_\alpha}e^{ik'_\alpha x'_\alpha}e^{ik'_\beta x'_\beta}(-\partial_\alpha^2+m^2)(-\partial_\alpha^2+m^2)(-\partial_\alpha^{2\prime}+m^2)(-\partial_\beta^{2\prime}+m^2)$$

$$\delta^{(4)}(x_\alpha-x'_\beta)\delta^{(4)}(x_\alpha-x'_\gamma)\delta^{(4)}(x'_\alpha-x'_\gamma{}')\delta^{(4)}(x'_\beta-x_\gamma)$$

(3.4)

We can ignore the other exponential integrals because, just like for the time ordered term, they do not alter the physics but only change the phase. Another check of this claim would be to note that the exponential Integrals are also the same. The partial derivatives also drop in the evaluation of the integrals.

So we can now look at the delta functions themselves:

$$\delta^{(4)}(x_\alpha - x'_\beta)\delta^{(4)}(x_\alpha - x'_\gamma)\delta^{(4)}(x'_\alpha - x'_\gamma{}')\delta^{(4)}(x'_\beta - x_\gamma) \qquad (3.6)$$

It is probably obvious that we must move to momentum space in order to make a comparison to the time ordered term. This can be achieved by invoking a Fourier transformation of our integrals. However it is important to note that doing so in this case will not change the sign of our delta function. This is because when we wrote out our initial expression we kept the sign of the exponentials to be positive. Since we did not invoke the hermitian conjugate of the exponentials at that stage, doing a fourier transform now simply reverts the exponentials to that form. Therefore the signs of the delta functions stay the same, as they must, and we can make a smooth transition to momentum space.

Our delta functions become:

$$\delta^{(4)}(p_\alpha - p'_\beta)\delta^{(4)}(p_\alpha - p'_\gamma)\delta^{(4)}(p'_\alpha - p'_\gamma)\delta^{(4)}(p'_\beta - p_\gamma) \qquad (3.7)$$

Which are the same delta functions for interactions that we computed in the time ordered term.

Footnote: **We use the same notation as before for the normalization constants, and all the bounds of integration are over all space*

4. Conclusions/ Further Research

My initial intuition suggested to me that all interactions that would arise in this theory would be completely independent of one another. It turned out that instead we had one particle, namely the alpha particle, Engaging in multiple interactions. The fact that all 6 particles engaged in at least one interaction is a a sign that the theory predicts that not all of the interactions would happen independently. This comes as a big surprise and offers some interesting questions. Why did the interactions turn out in this configuration, is this completely arbitrary? What are the consequences of having one particle engage in multiple Interactions? All of the particles are the same mass, so does this come into the story? What happens to the particles after the interaction insues? Why do only four interactions arise and not six like our intuition would dictate? These are questions that need to be answered and of course a deeper development of Quantum Field Theory should answer them with more precision and detail.

Citations:

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