

For notational simplicity in the imperfect detectability case, let $m_i = \sum_{j=1}^{M_i} z_{ij}$ denote the number of objects detected in unit i , so that the sample sum of the u_{ij} for unit i may be written $u_i = \sum_{j=1}^{m_i} y_{ij}/g_{ij}$ instead of the more formal notation with the indicator variable z_{ij} . Then, with probability of detection g_{ij} for the j th object of the i th unit, an unbiased estimator of τ by the results of this section is [cf. (9.18)]

$$\hat{\tau} = \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} \frac{y_{ij} z_{ij}}{g_{ij}} = \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} \frac{y_{ij}}{g_{ij}} = \sum_{i=1}^v \frac{u_i}{\pi_i}, \quad (9.23)$$

with variance given by (9.20), namely

$$\text{var}[\hat{\tau}] = E_z \left[\sum_{i=1}^v \sum_{i'=1}^v u_i u_{i'} \left(\frac{\pi_{ii'} - \pi_i \pi_{i'}}{\pi_i \pi_{i'}} \right) \right] + \sum_{i=1}^v \sum_{j=1}^{m_i} \frac{1 - g_{ij}}{g_{ij}} y_{ij}^2. \quad (9.24)$$

An unbiased variance estimator of this follows from (9.22), namely

$$\widehat{\text{var}}[\hat{\tau}] = v_1 + v_2, \quad (9.25)$$

where

$$\begin{aligned} v_1 &= v_0(\mathbf{u}_s) \\ &= \sum_{i=1}^v \sum_{i'=1}^v u_i u_{i'} \left(\frac{\pi_{ii'} - \pi_i \pi_{i'}}{\pi_i \pi_{i'}} \right) \end{aligned}$$

and

$$\begin{aligned} v_2 &= \hat{\tau}_0(\mathbf{t}_s) \\ &= \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} \frac{1 - g_{ij}}{g_{ij}^2} y_{ij}^2. \end{aligned}$$

The estimator (9.23) was given by Steinhurst and Samuel [1989, Equation 1], who used the alternative decomposition of the variance (9.7) instead of (9.19) to obtain

$$\text{var}[\hat{\tau}] = V_s + V_d, \quad (9.26)$$

where

$$V_s = \sum_{i=1}^v \sum_{i'=1}^v \tau_i \tau_{i'} \left(\frac{\pi_{ii'} - \pi_i \pi_{i'}}{\pi_i \pi_{i'}} \right) \quad (9.27)$$

and

$$V_d = \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} \frac{1 - g_{ij}}{g_{ij}} y_{ij}^2 = \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} v_{ij}. \quad (9.28)$$

Equations (9.24) and (9.26) are equivalent expressions for the variance of $\hat{\tau}$, though the individual terms differ. The alternative decomposition (9.26), in which conditional expectations with respect to the distribution of the detectability vector \mathbf{z} are first taken given a sample s and then unconditional expectations are taken with respect to the sampling design, is feasible here because, with a conventional design, the sampling design probabilities do not depend on the detections (as is the case with an adaptive design). The alternative decomposition (9.26) may be useful for the interpretation of the variance as the sum of a component V_s due to sampling and a component V_d due to detectability.

An unbiased estimate of the detectability component of variance V_d is provided by

$$v_d = \sum_{i=1}^v \frac{1}{\pi_i^2} \sum_{j=1}^{m_i} \frac{v_{ij}}{g_{ij}} = \sum_{i=1}^v \frac{1}{\pi_i^2} \sum_{j=1}^{m_i} \frac{1 - g_{ij}}{g_{ij}^2} y_{ij}^2.$$

This follows by using the fact that

$$\hat{\tau} = \sum_{i=1}^v \frac{1}{\pi_i} \sum_{j=1}^{m_i} \frac{y_{ij}}{g_{ij}}$$

of (9.23) is an unbiased estimate of

$$\tau = \sum_{i=1}^v \sum_{j=1}^{m_i} y_{ij}$$

and then by replacing y_{ij} by v_{ij}/π_i . An unbiased estimate of the sampling component of variance V_s is

$$v_s = \sum_{i=1}^v \sum_{i'=1}^v u_i u_{i'} \left(\frac{\pi_{ii'} - \pi_i \pi_{i'}}{\pi_i \pi_{i'}} \right) - \sum_{i=1}^v \left(\frac{1}{\pi_i^2} - \frac{1}{\pi_i} \right) \sum_{j=1}^{m_i} \frac{1 - g_{ij}}{g_{ij}^2} y_{ij}^2,$$

which is obtained by subtracting v_d from the unbiased variance estimator $\widehat{\text{var}}[\hat{\tau}]$ of (9.25).

Unfortunately, the alternative decomposition (9.26) does not facilitate the search for an estimator of variance as does (9.24). For an estimator of $\text{var}[\hat{\tau}]$, Steinhurst and Samuel [1989] gave $v^* = v_1 + v_d$. Although v_d is unbiased for V_d , the estimator v^* , unlike (9.25), is not unbiased for the actual variance [(9.24) or (9.26)] of the