

Analysis on manifolds. Exercise 1.

- (1) Let $M \subset \mathbb{R}^k$ be a smooth manifold.
 - a) Show that any connected component of M is a smooth manifold.
 - b) Show that if M is compact than any connected component of M is also compact.
- (2) Show that any manifold can be presented as an increasing countable union of compact subsets.
- (3) Let $f(z)$ be a holomorphic function considered as a smooth map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Show that the set of critical points of this map is equal to the set of zeros of the derivative f' .
- (4) Show that the union of two axes in \mathbb{R}^2 is not a manifold.
- (5) Let X be an m -manifold with boundary (defined in the class).
 - a) Show that the boundary ∂X is well defined.
 - b) Show that ∂X is an $(m - 1)$ -manifold (without boundary).
 - c) Show that $X \setminus \partial X$ is an m -manifold (without boundary).
- (6) Let M^m be a manifold. Let $x \in M$. Let $f_1, \dots, f_k: M \rightarrow \mathbb{R}$ be smooth functions such that their differentials at x are linearly independent. Show that there exist smooth functions f_{k+1}, \dots, f_m such that the map $F: X \rightarrow \mathbb{R}^m$ given by $F = (f_1, \dots, f_m)$ is a coordinate system on an open neighborhood U of x (namely $F(U) \subset \mathbb{R}^m$ is open and $F: U \rightarrow F(U)$ is a diffeomorphism).
- (7) Show that diffeomorphic manifolds have the same dimension.