We want to find the area of the parabolic segment bounded by the x-axis, the y-axis, the line $x = b$, and the curve $y = ax^m + c$, where $m \in \mathbb{Z}^+$ and $a, c > 0$. To do this we divide the base into n equal parts and at each point kb/n, $k = 1, 2, \ldots, n$, we construct an outer rectangle of height $a(kb/n)^m + c$. Similarly at each point kb/n, $k = 1, 2, \ldots, n-1$, we construct an inner rectangle of height $a(kb/n)^m + c$. The area of the parabolic segment is between the sum s_n of the areas of the inner rectangles and the sum S_n of the areas of the outer rectangles. The diagram shows the area of the typical outer rectangle.

$$s_n = \sum_{k=1}^{n-1} \left(\frac{ab^{m+1}}{n^{m+1}} k^m + \frac{bc}{n} \right) = \frac{ab^{m+1}}{n^{m+1}} \sum_{k=1}^{n-1} (k^m) + \frac{(n-1)cb}{n}$$

$$S_n = \sum_{k=1}^{n} \left(\frac{ab^{m+1}}{n^{m+1}} k^m + \frac{bc}{n} \right) = \frac{ab^{m+1}}{n^{m+1}} \sum_{k=1}^{n} (k^m) + cb$$

We use a previously proved theorem which states that

$$\sum_{k=1}^{n-1} (k^m) < \frac{n^{m+1}}{m+1} < \sum_{k=1}^{n} (k^m) \tag{1}$$

and multiply by and add the appropriate terms in order to obtain

$$s_n < \frac{ab^{m+1}}{m+1} + \frac{(n-1)cb}{n} < \frac{ab^{m+1}}{m+1} + cb$$

from the left side and

$$S_n > \frac{ab^{m+1}}{m+1} + cb$$

from the right side so that, combined, we have

$$s_n < \frac{ab^{m+1}}{m+1} + cb < S_n.$$

This is the first part of the proof. The second part is to show that, if A is any number satisfying $s_n < A < S_n$ for all $n \in \mathbb{Z}^+$, then $A = \frac{ab^{m+1}}{m+1} + cb$. So assume $s_n < A < S_n$.

Adding n^m to the left two sides of (1) and subtracting n^m from the right two sides, we can get

$$\sum_{k=1}^{n} (k^m) < \frac{n^{m+1}}{m+1} + n^m$$

$$\sum_{k=1}^{n-1} (k^m) > \frac{n^{m+1}}{m+1} - n^m$$

and, by multiplying by and adding the appropriate terms,

$$S_n < \frac{ab^{m+1}}{m+1} + cb + \frac{ab^{m+1}}{n}$$

and

$$s_n > \frac{ab^{m+1}}{m+1} + \frac{(n-1)cb}{n} - \frac{ab^{m+1}}{n}.$$

The trouble arises here because at this point I would like to show that $s_n > \frac{ab^{m+1}}{m+1} + cb - \frac{ab^{m+1}}{n}$ so that I could have the combined inequality

$$\frac{ab^{m+1}}{m+1} + cb - \frac{ab^{m+1}}{n} < s_n < A < S_n < \frac{ab^{m+1}}{m+1} + cb + \frac{ab^{m+1}}{n}$$

and then use this to show that $A \neq \frac{ab^{m+1}}{m+1} + cb$ implies a bound for n. This was the way the sample problem (without a constant term) in the book went about it.