Each transformation group of a set naturally acts on that set ($T_g \equiv g$), but may also act on other sets. For example, consider an equilateral triangle. The group of its six isometries acts on the set of its two orientations: the reflections reverse the orientation, the rotations do not.

Problem 6. Which permutations of the three coordinate axes are realized by the action of the group of isometries of the cube $\max(|x|, |y|, |z|) \leq 1$ on the set of axes?

Answer. All six.

Problem 7. How does the group of linear changes of coordinates act on the set of matrices of linear operators from a space into itself?

Answer. $T_g m = gmg^{-1}$.

The transformation T_g is also called the *action of the element* g of the group G on M. The action of the group G on M defines another mapping $T : G \times M \to M$ assigning to the pair $g \in G$, $m \in M$ the point $T_g m$.

If the action T is fixed, then the result $T_g m$ of the action of the element g of the group G on a point m of the set M is denoted by gm for short. Thus $(fg)m = f(gm)$, and so the parentheses are usually omitted.

Let us fix a point m of the set M and act on it by all the elements of the group G. We thereby obtain a subset $\{gm, \ g \in G\}$ of the set M. This subset is called the *orbit of the point* m (for the given group action), and is denoted Gm.

Problem 8. Find the orbits of the group of rotations of the plane about zero.

Problem 9. Prove that any two orbits of an action are either disjoint or coincident.

Problem 10. How many orbits are there in the action of the group of isometries of the tetrahedron on the set of unordered pairs of its edges?

Problem 11. How many colorings of the six faces of a cube by six colors $1, \ldots, 6$ are essentially different (cannot be transformed into one another by rotations of the cube)?

Answer. $6!/24 = 30$.

A mapping $\varphi : G \to H$ of the group G into the group H is called a *homomorphism* if it takes products into products and inverses into inverses:

$$\varphi(fg) = \varphi(f)\varphi(g); \quad \varphi(g^{-1}) = (\varphi(g))^{-1}.$$

The action of a group G on a set M is a homomorphism of the group G into the group of all transformations of the set M.