Interference

For our next topic, we're going to study a phenomenon that might appear to be distinctly different from what one encounters in classical physics. This phenomenon, known as quantum-theoretic *interference*, turns out to resemble the interference of classical waves in some ways, but also differs from the interference of classical waves in other ways.

Quantum-theoretic interference is easiest to introduce through a famous thought experiment, known as the *double-slit experiment*. The idea is to send particles—one at a time—toward a wall with two small slits or holes in it, where the slits are assumed to be close together, and then record where the particles eventually land on a detection screen on the other side of the wall. Broadly speaking, we can let A be a random variable that denotes each particle's initial conditions, we can let B denote which slit the particle goes through, and we can let C denote where the particle ends up on the detection screen.

If one carries out this experiment using small stones, then one finds a distribution of landing sites C on the detection screen that's concentrated at the center of the detection screen, at a point aligned with the midpoint between the two slits, and is less concentrated the farther away from the center one looks. One can provide an intuitive explanation for this distribution of landing sites on the detection screen. A few stones make it through the upper slit in the wall, and lead to a concentration of landing sites on the detection screen lined up with the first slit. A few stones instead go through the lower slit in the wall, leading to a concentration of landing sites aligned with the second slit. Because the two slits are close together, the two concentrations on the detection screen merge into one larger distribution centered between the two slits.

If one instead carries out this experiment repeatedly using electrons, sending just one electron each time, then one finds that the distribution of landing sites C features a strange assortment of peaks and valleys. Near the center of the detection screen, there is a high concentration of landing sites, but slightly off-center, there are no landing sites. Slightly farther out, there is another concentration of landing sites, but then, even farther out, none. This pattern of peaks and valleys continues, with the peaks getting smaller and smaller the farther one looks away from the center of the detection screen.

This pattern of landing sites C, recorded over many trials, resembles the crests and troughs that would arise from waves approaching the two slits, spreading or diffracting from each slit, and then exhibiting constructive and destructive interference on the other side. This resemblance between the pattern of the electron's landing sites over many trials and the shape of interfering waves has suggested to some people that an electron might itself have a wavelike nature or even *be* a wave in some circumstances—a notion known as *wave-particle duality*.

One crucial difference, of course, is that for actual waves, the pattern of crests and troughs is explicitly manifest all at once in the observable shape of the wave as a whole. When you look at, say, waves on a lake, you can see the crests and troughs quite directly. By contrast, for an electron going through a double-slit experiment, the interference pattern shows up only point by point, over the course of sending electrons one at a time over many trials. We don't directly see a wave at all in the electron case—we only see *indirect* evidence of a wave through the distribution of landing sites over many trials, like the formation of a painting from a pointillist technique.

If there is, in fact, a wave involved in this story, then what is the nature of this wave? We've already seen that a quantum system can have an associated *wave function* (17.12). Indeed, the wavelike behavior of an electron in the double-slit experiment is usually associated with the electron's wave function. The trouble is that a wave function doesn't live in *physical three-dimensional space*, but in the system's *configuration space*, meaning an abstract space whose points formally denote the system's distinct possible configurations or snapshots. Recall that for a single particle in physical three-dimensional space, the relevant configuration space is itself three-dimensional, so it's easy to confuse the two kinds of spaces. But physical space and configuration space are conceptually different things, and this difference between them becomes manifest if we were to imagine sending *two* electrons at a time, instead of just one. Then because it takes six numbers to specify the locations of two particles, the configuration space would now be six-dimensional, rather than three-dimensional, meaning that the corresponding wave function would reside in a six-dimensional configuration space. (Technically, one needs to modify the configuration space slightly to account for the fact that electrons are indistinguishable particles.) Although it's possible to try to take configuration spaces of arbitrary dimensions seriously as the seat of physical reality, as Schrödinger himself argued in his early papers on wave mechanics, and as is advocated in an interpretative framework known today as *wave-function realism*, that metaphysical posture might be a hard sell for many people.

There's a slight modification of the double-slit experiment that makes its behavior even stranger. Suppose that we add an indicator qubit just behind the two slits, where this indicator qubit changes its configuration based on which slit the electron goes through. This qubit has just two configurations, which we'll denote by $\uparrow$ and $\downarrow$, and let's assume that it always starts off in its $\uparrow$ configuration at the beginning of each trial. Suppose that the indicator qubit ends up in its $\uparrow$ configuration if the electron goes through the upper slit, and ends up in its $\downarrow$ configuration if the electron goes through the lower slit.

If this indicator qubit is included in each trial of the experiment, then the distribution of the electron's landing sites C over many trials will look more like for the case of small stones, without all those striking peaks and valleys. However, if at some point during the flight of each electron from the wall to the detection screen, the indicator qubit is programmed to erase its memory, so that it always returns to its $\uparrow$ configuration regardless of its previous configuration during the electron's transit to the detection screen, then the peaks and valleys in the distribution of landing sites will reappear.

Treating the electron as a closed system, let's see how these strange behaviors naturally emerge from the electron's unistochastic dynamics. To make our preliminary treatment easier to follow, let's simplify or coarse-grain the electron's configuration space so that we can model it as a qubit with just two possible configurations: 1 and 2. We'll take 1 to mean that the coarse-grained electron is above the middle line of the apparatus, corresponding both to the upper slit and also to the upper portion of the detection screen, and we'll take 2 to mean that the coarse-grained electron is below the middle line of the apparatus, corresponding both to the lower slit and also to the lower portion of the detection screen.

The coarse-grained electron has a 2×2 unistochastic matrix

$$\mathbf{\Gamma}(t) = \begin{pmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{pmatrix}.$$

The entries of this stochastic matrix are expressible in terms of corresponding entries of a 2×2 unitary matrix $\mathbf{U}(t)$,

$$\mathbf{\Gamma}(t) = \begin{pmatrix} |U_{11}(t)|^2 & |U_{12}(t)|^2 \\ |U_{21}(t)|^2 & |U_{22}(t)|^2 \end{pmatrix},$$

where

$$\mathbf{U}(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix}.$$

Because $\mathbf{U}(t)$ is unitary, we know from (20.3) that it has an inverse matrix given by its adjoint:

$$\boxed{\mathbf{U}^\dagger(t) = \mathbf{U}^{-1}(t).}$$

Let's consider the product of $\mathbf{U}(t)$ itself with the adjoint of $\mathbf{U}(t')$ evaluated at another time t':

$$\mathbf{U}(t)\mathbf{U}^\dagger(t').$$

The resulting matrix is again unitary, and is a function of both t and t'. As in (20.10), we call it $\mathbf{U}(t \leftarrow t')$:

$$\boxed{\mathbf{U}(t \leftarrow t') \equiv \mathbf{U}(t)\mathbf{U}^\dagger(t').}$$

As we saw in deriving (20.11), we see that we can rearrange this definition to obtain the following *composition law*:

$$\boxed{\underbrace{\mathbf{U}(t)}_{0 \text{ to } t} = \mathbf{U}(t \leftarrow t')\,\underbrace{\mathbf{U}(t')}_{0 \text{ to } t'}.}$$

In detail, from the rules of matrix multiplication, we have

$$U_{ij}(t) = \sum_{k=1}^{2} U_{ik}(t \leftarrow t')U_{kj}(t'),$$

so this composition law consists of the following four equations:

$$\begin{aligned}
U_{11}(t) &= U_{11}(t \leftarrow t')U_{11}(t') + U_{12}(t \leftarrow t')U_{21}(t'),\\
U_{12}(t) &= U_{11}(t \leftarrow t')U_{12}(t') + U_{12}(t \leftarrow t')U_{22}(t'),\\
U_{21}(t) &= U_{21}(t \leftarrow t')U_{11}(t') + U_{22}(t \leftarrow t')U_{21}(t'),\\
U_{22}(t) &= U_{21}(t \leftarrow t')U_{12}(t') + U_{22}(t \leftarrow t')U_{22}(t').
\end{aligned}$$

The matrix $\mathbf{U}(t \leftarrow t')$ is unitary, so squaring its entries $|U_{ij}(t \leftarrow t')|^2$ defines a 2×2 unistochastic matrix $\mathbf{\Gamma}(t \leftarrow t')$:

$$\boxed{\mathbf{\Gamma}(t \leftarrow t') = \begin{pmatrix} \Gamma_{11}(t \leftarrow t') & \Gamma_{12}(t \leftarrow t') \\ \Gamma_{21}(t \leftarrow t') & \Gamma_{22}(t \leftarrow t') \end{pmatrix} \equiv \begin{pmatrix} |U_{11}(t \leftarrow t')|^2 & |U_{12}(t \leftarrow t')|^2 \\ |U_{21}(t \leftarrow t')|^2 & |U_{22}(t \leftarrow t')|^2 \end{pmatrix}.} \tag{22.1}$$

However, this new unistochastic matrix $\mathbf{\Gamma}(t \leftarrow t')$ does *not* generally lead to a corresponding composition law for the system's original unistochastic matrix $\mathbf{\Gamma}(t)$. That is, generically speaking, we'll find that

$$\boxed{\mathbf{\Gamma}(t) \neq \mathbf{\Gamma}(t \leftarrow t')\mathbf{\Gamma}(t'),} \tag{22.2}$$

so despite the notation for $\mathbf{\Gamma}(t \leftarrow t')$, we cannot say (at this point, at least) that it has a clear physical interpretation of describing stochastic dynamics from t' to t. Fundamentally, this discrepancy occurs because taking absolute-value-squares of individual entries of matrices is not compatible with matrix multiplication. We'll now show that this discrepancy between the true, indivisible dynamics expressed by $\mathbf{\Gamma}(t)$ on the left-hand side of (22.2) and the incorrect, divisible dynamics expressed by $\mathbf{\Gamma}(t \leftarrow t')\mathbf{\Gamma}(t')$ on the right-hand side is precisely why interference occurs.

Let's compute the discrepancy between the left-hand side and right-hand side of (22.2) explicitly for our coarse-grained, two-configuration electron, treating $t = 0$ as the launch time, t' as the time the electron reaches the wall with the slits in it, and t as the time when the electron lands on the detection screen. For definiteness, let's focus on $\Gamma_{11}(t) \equiv p(1, t|1, 0)$, the conditional probability for the electron to be above the middle line of the double-slit set-up at the detection time t, given that the electron was already above the middle line at the initial time $t = 0$, again within the context of our coarse-graining of the electron's configuration space down to just two possible configurations.

The corresponding entry of the right-hand side of (22.2) is, from the standard rules of matrix multiplication, given by

$$[\Gamma(t \leftarrow t')\Gamma(t')]_{11} = \Gamma_{11}(t \leftarrow t')\Gamma_{11}(t') + \Gamma_{12}(t \leftarrow t')\Gamma_{21}(t').$$

Inserting the basic relationships $\Gamma_{ik}(t \leftarrow t') = |U_{ik}(t \leftarrow t')|^2$ and $\Gamma_{jk}(t') = |U_{jk}(t')|^2$, we have

$$[\Gamma(t \leftarrow t')\Gamma(t')]_{11} = |U_{11}(t \leftarrow t')|^2|U_{11}(t')|^2 + |U_{12}(t \leftarrow t')|^2|U_{21}(t')|^2. \tag{22.3}$$

In a sense, this is the predicted conditional probability based on the assumption that we can divide up the electron's path into two simple stochastic processes: a unistochastic process from 0 to t' and then, separately, another unistochastic process from t' to t.

Computing the left-hand side of (22.2), meaning the electron's *actual* conditional probability, we find

$$\Gamma_{11}(t) = |U_{11}(t)|^2.$$

Invoking the composition law $\mathbf{U}(t) = \mathbf{U}(t \leftarrow t')\mathbf{U}(t')$ from (20.11), we have

$$\Gamma_{11}(t) = |U_{11}(t \leftarrow t')U_{11}(t') + U_{12}(t \leftarrow t')U_{21}(t')|^2. \tag{22.4}$$

Comparing (22.3) and (22.4), we see that we have, schematically speaking,

$$|u|^2 + |w|^2 \quad \text{versus} \quad |u + w|^2.$$

These expressions are indeed not in agreement. Specifically,

$$\begin{aligned}|u+w|^2 &= \overline{(u+w)}(u+w)\\ &= (\overline{u}+\overline{w})(u+w)\\ &= \overline{u}u+\overline{u}w+\overline{w}u+\overline{w}w\\ &= |u|^2+\overline{u}w+\overline{w}u+|w|^2.\end{aligned}$$

The discrepancy therefore lies in the two cross terms $\overline{u}w$ and $\overline{w}u$, which, despite being complex-valued individually, add up to a real-valued quantity, because for any complex number $z \equiv a+ib$, one has $z+\overline{z} = (a+ib)+(a-ib) = 2a$.

Carrying out the multiplication in (22.4) explicitly, we have

$$\begin{aligned}\Gamma_{11}(t) &= \overline{\Bigg[\underbrace{U_{11}(t \leftarrow t')U_{11}(t')}_{u} + \underbrace{U_{12}(t \leftarrow t')U_{21}(t')}_{w}\Bigg]}\\ &\quad \times \Bigg[\underbrace{U_{11}(t \leftarrow t')U_{11}(t')}_{u} + \underbrace{U_{12}(t \leftarrow t')U_{21}(t')}_{w}\Bigg]\\ &= \underbrace{|U_{11}(t \leftarrow t')|^2|U_{11}(t')|^2}_{|u|^2}\\ &\quad + \underbrace{\overline{U_{11}(t \leftarrow t')U_{11}(t')}}_{\overline{u}}\,\underbrace{U_{12}(t \leftarrow t')U_{21}(t')}_{w}\\ &\quad + \underbrace{\overline{U_{12}(t \leftarrow t')U_{21}(t')}}_{\overline{w}}\,\underbrace{U_{11}(t \leftarrow t')U_{11}(t')}_{u}\\ &\quad + \underbrace{|U_{11}(t \leftarrow t')|^2|U_{11}(t')|^2}_{|w|^2}.\end{aligned}$$

The difference between this expression for $\Gamma_{11}(t)$ and (22.3) for $[\Gamma(t \leftarrow t')\Gamma(t')]_{11}$ consists precisely of the two cross terms:

$$\left.\begin{aligned}\Gamma_{11}(t) - [\Gamma(t \leftarrow t')\Gamma(t')]_{11} &= \overline{U_{11}(t \leftarrow t')U_{11}(t')}U_{12}(t \leftarrow t')U_{21}(t')\\ &\quad + \overline{U_{12}(t \leftarrow t')U_{21}(t')}U_{11}(t \leftarrow t')U_{11}(t').\end{aligned}\right\} \tag{22.5}$$

Using a state vector $\mathbf{\Psi}(t')$ as in (17.16) to denote the first column of the time-evolution operator $\mathbf{U}(t')$, we can write (22.3) somewhat more succinctly as

$$[\Gamma(t \leftarrow t')\Gamma(t')]_{11} = |U_{11}(t \leftarrow t')|^2|\Psi_1(t')|^2 + |U_{12}(t \leftarrow t')|^2|\Psi_2(t')|^2, \tag{22.6}$$

and we can write the cross terms (22.5) as

$$\begin{aligned}\Gamma_{11}(t) - [\Gamma(t \leftarrow t')\Gamma(t')]_{11} &= \overline{U_{11}(t \leftarrow t')\Psi_1(t')}U_{12}(t \leftarrow t')\Psi_2(t')\\ &\quad + \overline{U_{12}(t \leftarrow t')\Psi_2(t')}U_{11}(t \leftarrow t')\Psi_1(t'),\end{aligned}$$

where the right-hand side is the sum of a complex-valued quantity with its complex conjugate, and is therefore real-valued. Recalling the simplest version of the Born rule, (17.19), we have that $p_k(t') = |\Psi_k(t')|^2$ is the standalone probability for the electron to be in its kth configuration at the time t', so we can recast (22.6) as

$$[\Gamma(t \leftarrow t')\Gamma(t')]_{11} = \Gamma_{11}(t \leftarrow t')p_1(t') + \Gamma_{12}(t \leftarrow t')p_2(t'),$$

meaning that we can write $\Gamma_{11}(t)$ as a whole as

$$\boxed{\begin{aligned}\Gamma_{11}(t) &= \Gamma_{11}(t \leftarrow t')p_1(t') + \Gamma_{12}(t \leftarrow t')p_2(t')\\ &\quad + \overline{U_{11}(t \leftarrow t')\Psi_1(t')}U_{12}(t \leftarrow t')\Psi_2(t')\\ &\quad + \overline{U_{12}(t \leftarrow t')\Psi_2(t')}U_{11}(t \leftarrow t')\Psi_1(t').\end{aligned}} \tag{22.7}$$

The first line of this equation looks like a simple marginalization rule at t', but the other two lines make clear that this marginalization rule at t' is not the whole story, due to the indivisibility of the unistochastic dynamics.

For example, if

$$\Psi_1(t') \equiv U_{11}(t') = \frac{1}{\sqrt{2}} e^{i\alpha}, \qquad \Psi_2(t') \equiv U_{21}(t') = \frac{1}{\sqrt{2}} e^{i\beta},$$

so that the electron has equal probability of going through either slit,

$$p_1(t') = |\Psi_1(t')|^2 = \frac{1}{2}, \qquad p_2(t') = |\Psi_2(t')|^2 = \frac{1}{2},$$

and if

$$U_{11}(t \leftarrow t') = a e^{i\theta}, \qquad U_{12}(t \leftarrow t') = b e^{i\phi},$$

where a and b are real-valued and non-negative, so that

$$\Gamma_{11}(t \leftarrow t') = |U_{11}(t \leftarrow t')|^2 = a^2, \qquad \Gamma_{12}(t \leftarrow t') = |U_{12}(t \leftarrow t')|^2 = b^2,$$

then

$$\Gamma_{11}(t \leftarrow t') p_1(t') + \Gamma_{12}(t \leftarrow t') p_2(t') = \frac{1}{2} a^2 + \frac{1}{2} b^2,$$

and the interference cross-terms are

$$\begin{aligned}
&\overline{U_{11}(t \leftarrow t')\Psi_1(t')} U_{12}(t \leftarrow t')\Psi_2(t') \\
&\quad + \overline{U_{12}(t \leftarrow t')\Psi_2(t')} U_{11}(t \leftarrow t')\Psi_1(t') \\
&= \overline{a e^{i\theta} \frac{1}{\sqrt{2}} e^{i\alpha}} b e^{i\phi} \frac{1}{\sqrt{2}} e^{i\beta} + \overline{b e^{i\phi} \frac{1}{\sqrt{2}} e^{i\beta}} a e^{i\theta} \frac{1}{\sqrt{2}} e^{i\alpha} \\
&= \frac{1}{2} a e^{-i\theta} e^{-i\alpha} b e^{i\phi} e^{i\beta} + \frac{1}{2} b e^{-i\phi} e^{-i\beta} a e^{i\theta} e^{i\alpha} \\
&= ab \frac{1}{2} \left[e^{i(\theta+\alpha-\phi-\beta)} + e^{-i(\theta+\alpha-\phi-\beta)} \right].
\end{aligned}$$

Using Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ from (12.35), as well as $e^{-i\theta} = \cos\theta - i\sin\theta$, this last expression becomes

$$\begin{aligned}
&ab \frac{1}{2} \left[\cos(\theta+\alpha-\phi-\beta) + \cancel{i \sin(\theta+\alpha-\phi-\beta)} + \cos(\theta+\alpha-\phi-\beta) - \cancel{i \sin(\theta+\alpha-\phi-\beta)} \right] \\
&= \frac{ab}{2} 2 \cos(\theta+\alpha-\phi-\beta) \\
&= ab \cos(\theta+\alpha-\phi-\beta).
\end{aligned}$$

Putting everything together, (22.7) becomes

$$\Gamma_{11}(t) = \frac{1}{2} a^2 + \frac{1}{2} b^2 + \cos(\theta+\alpha-\phi-\beta),$$

where the sinusoidal term at the end represents the interference. In general, the phase angles θ and ϕ will depend on the final time t, so different values of the final time will lead to slightly different amounts of interference. In particular, for some values of $\theta+\alpha-\phi-\beta$, the interference term will be negative, partially canceling the preceding two terms and reducing the probability for the electron to end up in its 1 configuration at t.

There was nothing special about the specific entry $\Gamma_{11}(t)$, or even the 2×2 case. In general, for the case of a system with N possible configurations, one finds that

$$\boxed{\begin{aligned} \Gamma_{ij}(t) = &\sum_{k=1}^{N} \Gamma_{ik}(t \leftarrow t') p_k(t') \\ &+ \sum_{k \neq l} \overline{U_{ik}(t \leftarrow t')\Psi_k(t')} U_{il}(t \leftarrow t')\Psi_l(t'), \end{aligned}} \tag{22.8}$$

where the first line looks like simple marginalization at t', as though the unistochastic dynamics were divisible at t', and where the cross terms in the second line describe interference, and make clear that the actual unistochastic dynamics is, in fact, indivisible.

In this more general case, suppose again that

$$\Psi_1(t') = \frac{1}{\sqrt{2}} e^{i\alpha}, \qquad \Psi_2(t') = \frac{1}{\sqrt{2}} e^{i\beta},$$

so that the system is equally likely to be in its configurations 1 or 2 at t' (with zero probability of being in any other configuration at t'), but now let's let the final configuration $i = 1, \dots, N$ at t be more general, with

$$U_{i1}(t \leftarrow t') = a_i e^{i\theta_i}, \qquad U_{i2}(t \leftarrow t') = b_i e^{i\phi_i}.$$

Then

$$\Gamma_{i1}(t) = \frac{1}{2} a_i^2 + \frac{1}{2} b_i^2 + a_i b_i \cos(\theta_i + \alpha - \phi_i - \beta).$$

So some final configurations $i = 1, \dots, N$ will exhibit positive interference terms, meaning constructive interference, whereas other final configurations will exhibit negative interference terms, meaning destructive interference.

Decoherence and Division Events

Let's return to the double-slit experiment again, and formulate the interference in a slightly different way that will eventually smooth the way toward introducing an indicator qubit. Once more, let's suppose for simplicity that we've coarse-grained the electron's configuration space so that it consists of just two possible configurations: 1, for when the electron is in the upper portion of the apparatus, and 2, for when the electron is in the lower portion of the apparatus. Again for simplicity, suppose that the electron is launched from a definite configuration j at $t = 0$ (say, $j = 1$), that it arrives at the wall with the two slits at $t' > 0$, and that it lands on the detection screen at $t > t'$.

Let $p_1(t')$ be the standalone probability that the electron passes through the upper slit at t', and let $p_2(t')$ be the standalone probability that the electron passes through the lower slit at t':

$$\boxed{\begin{aligned} p_1(t') &\equiv p(\text{electron passes through the upper slit at } t'), \\ p_2(t') &\equiv p(\text{electron passes through the lower slit at } t'). \end{aligned}} \tag{22.9}$$

Because the electron has some specific configuration j at $t = 0$, the electron has a corresponding 2×1 state vector

$$\boxed{\boldsymbol{\Psi}(t) = \begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix},} \tag{22.10}$$

whose two entries at t' are

$$\boxed{\begin{aligned} &\Psi_1(t'), \\ &\Psi_2(t'). \end{aligned}} \tag{22.11}$$

From the simplest form of the Born rule in (17.19), these entries of the electron's state vector at t' are related to the corresponding standalone probabilities (22.9) according to

$$\boxed{\begin{aligned} p_1(t') &= |\Psi_1(t')|^2, \\ p_2(t') &= |\Psi_2(t')|^2. \end{aligned}} \tag{22.12}$$

At the detection time $t > t'$, the electron interacts with the detection screen, so the electron is no longer a closed system starting at the detection time. However, between the initial time $t = 0$ and the detection time $t > t'$, we can treat the electron as a closed system. The electron then has unistochastic dynamics from the initial time $t = 0$ all the way to the detection time $t > t'$, from which it follows that there exists a unitary operator $\mathbf{U}(t \leftarrow t')$, as defined in (20.10), that transforms the electron's state vector $\mathbf{\Psi}(t')$ at the wall time t' into its state vector $\mathbf{\Psi}(t)$ at the detection time $t > t'$:

$$\boxed{\mathbf{\Psi}(t) = \mathbf{U}(t \leftarrow t')\mathbf{\Psi}(t').} \tag{22.13}$$

That is,

$$\boxed{\begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix} = \begin{pmatrix} U_{11}(t \leftarrow t') & U_{12}(t \leftarrow t') \\ U_{21}(t \leftarrow t') & U_{22}(t \leftarrow t') \end{pmatrix} \begin{pmatrix} \Psi_1(t') \\ \Psi_2(t') \end{pmatrix}.} \tag{22.14}$$

Carrying out the matrix multiplication explicitly, we see that the two entries $\Psi_1(t)$ and $\Psi_2(t)$ of the electron's state vector at the detection time $t > t'$ are given by

$$\begin{aligned} \Psi_1(t) &= U_{11}(t \leftarrow t')\Psi_1(t') + U_{12}(t \leftarrow t')\Psi_2(t'), \\ \Psi_2(t) &= U_{21}(t \leftarrow t')\Psi_1(t') + U_{22}(t \leftarrow t')\Psi_2(t'). \end{aligned}$$

Let's give names to the four distinct terms appearing on the right-hand sides of these two equations:

$$\boxed{\begin{aligned} \Phi_1(t) &\equiv U_{11}(t \leftarrow t')\Psi_1(t'), \\ \Omega_1(t) &\equiv U_{12}(t \leftarrow t')\Psi_2(t'), \\ \Phi_2(t) &\equiv U_{21}(t \leftarrow t')\Psi_1(t'), \\ \Omega_2(t) &\equiv U_{22}(t \leftarrow t')\Psi_2(t'). \end{aligned}} \tag{22.15}$$

Then the two equations for $\Psi_1(t)$ and $\Psi_2(t)$ above take the simpler form

$$\boxed{\begin{aligned} \Psi_1(t) &= \Phi_1(t) + \Omega_1(t), \\ \Psi_2(t) &= \Phi_2(t) + \Omega_2(t). \end{aligned}} \tag{22.16}$$

Applying the Born rule (17.19) again, we see that the standalone probabilities for the electron at the detection time $t > t'$ are given by

$$\boxed{\begin{aligned} p_1(t) &= |\Psi_1(t)|^2 = |\Phi_1(t) + \Omega_1(t)|^2, \\ p_2(t) &= |\Psi_2(t)|^2 = |\Phi_2(t) + \Omega_2(t)|^2. \end{aligned}} \tag{22.17}$$

Notice here that in each expression on the right-hand side, we sum $\Phi_i(t) + \Omega_i(t)$ first, and then take the absolute-value-square of the result. Using $|z|^2 = \overline{z}z$, these expressions yield

$$\begin{aligned} p_1(t) &= \left(\overline{\Phi_1(t)} + \overline{\Omega_1(t)}\right)(\Phi_1(t) + \Omega_1(t)) \\ &= |\Phi_1(t)|^2 + |\Omega_1(t)|^2 + \overline{\Phi_1(t)}\Omega_1(t) + \overline{\Omega_1(t)}\Phi_1(t), \\ p_2(t) &= \left(\overline{\Phi_2(t)} + \overline{\Omega_2(t)}\right)(\Phi_2(t) + \Omega_2(t)) \\ &= |\Phi_2(t)|^2 + |\Omega_2(t)|^2 + \overline{\Phi_2(t)}\Omega_2(t) + \overline{\Omega_2(t)}\Phi_2(t). \end{aligned}$$

The cross-terms $\overline{\Phi_i(t)}\Omega_i(t) + \overline{\Omega_i(t)}\Phi_i(t)$ appearing in these formulas, just like the cross terms appearing in (22.7), represent interference.

We're now ready to introduce the indicator qubit. Again, in each run or trial of the experiment, this indicator qubit will sit between the two slits and record which slit the electron passes through. The indicator qubit will have two possible configurations: $\uparrow$, which is the indicator qubit's initial configuration and the one it will maintain if the electron goes through the upper slit, and $\downarrow$, which will be the indicator qubit's new configuration if the electron goes through the lower slit. For now, we'll assume for simplicity that the indicator qubit doesn't undergo any other time evolution.

The electron alone is no longer a closed system during the time interval from the initial time interval $t = 0$ to the detection time $t > t'$, but will be assumed to interact with the indicator qubit *slightly before* the wall time $t' > 0$. However, we can treat the *composite* system consisting of the electron *together* with the indicator qubit as a closed system from the initial time $t = 0$ to the detection time $t > t'$. This composite system has a configuration space consisting of four configurations, which we'll denote by the following ordered pairs:

$$\boxed{\begin{aligned}&(1, \uparrow),\\ &(2, \uparrow),\\ &(1, \downarrow),\\ &(2, \downarrow).\end{aligned}} \tag{22.18}$$

In each of these ordered pairs, the first label denotes the configuration of our coarse-grained electron, and the second label denotes the configuration of the indicator qubit.

By assumption, at the wall time t', the composite system has zero probability of being in the configurations $(1, \downarrow)$ or $(2, \uparrow)$. Let's suppose that the joint probability $p_{1,\uparrow}(t')$ for the composite system to be in the configuration $(1, \uparrow)$ at t' has the same value $p_1(t')$ as in (22.9), and, similarly, that the joint probability $p_{2,\downarrow}(t')$ for the composite system to be in the configuration $(2, \downarrow)$ at t' has the same value $p_2(t')$ as in (22.9). Altogether, we then have

$$\boxed{\begin{aligned}&p_{1,\uparrow}(t') = p_1(t'),\\ &p_{2,\uparrow}(t') = 0,\\ &p_{1,\downarrow}(t') = 0,\\ &p_{2,\downarrow}(t') = p_2(t').\end{aligned}} \tag{22.19}$$

The composite system has a 4×1 state vector

$$\boxed{\mathbf{\Psi}(t) = \begin{pmatrix}\Psi_{1,\uparrow}(t)\\ \Psi_{2,\uparrow}(t)\\ \Psi_{1,\downarrow}(t)\\ \Psi_{2,\downarrow}(t)\end{pmatrix}.} \tag{22.20}$$

Its individual entries at the wall time t' are given by

$$\boxed{\begin{aligned}&\Psi_{1,\uparrow}(t') = \Psi_1(t'),\\ &\Psi_{2,\uparrow}(t') = 0,\\ &\Psi_{1,\downarrow}(t') = 0,\\ &\Psi_{2,\downarrow}(t') = \Psi_2(t'),\end{aligned}} \tag{22.21}$$

where, from the Born rule (17.19), these entries are related to the composite system's joint probabilities (22.19) at the wall time t' according to

$$\boxed{\begin{aligned}&p_{1,\uparrow}(t') = |\Psi_{1,\uparrow}(t')|^2 = p_1(t'),\\ &p_{2,\uparrow}(t') = |\Psi_{2,\uparrow}(t')|^2 = 0,\\ &p_{1,\downarrow}(t') = |\Psi_{1,\downarrow}(t')|^2 = 0,\\ &p_{2,\downarrow}(t') = |\Psi_{2,\downarrow}(t')|^2 = p_2(t').\end{aligned}} \tag{22.22}$$

By assumption, the electron and the indicator qubit interact with each other just before the wall time $t' > 0$, but do not interact with anything else from the wall time $t' > 0$ to the detection time $t > t'$. During this later time interval, the two subsystems therefore have their own individual unitary time-evolution operators. Thinking of our electron as the *subject system* $\mathcal{S}$ under study, and thinking of the indicator qubit as the simplest possible model of an *environment* $\mathcal{E}$, the unitary time-evolution operator $\mathbf{U}^{\mathcal{SE}}(t \leftarrow t')$ for the composite system $\mathcal{SE}$ factorizes as

$$\boxed{\mathbf{U}^{\mathcal{SE}}(t \leftarrow t') = \mathbf{U}^{\mathcal{S}}(t \leftarrow t') \otimes \mathbf{U}^{\mathcal{E}}(t \leftarrow t'),} \tag{22.23}$$

where $\mathbf{U}^{\mathcal{S}}(t \leftarrow t')$ describes the time evolution of the electron $\mathcal{S}$ from t' to t, and where $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$ describes the time evolution of the indicator qubit from t' to t. Here the tensor-product symbol $\otimes$ is merely notational shorthand for

$$\boxed{U^{\mathcal{SE}}_{(i,e),(i',e')}(t \leftarrow t') = U^{\mathcal{S}}_{ii'}(t \leftarrow t')U^{\mathcal{E}}_{ee'}(t \leftarrow t'),} \tag{22.24}$$

with $i = 1, 2$ labeling the two possible configurations of the electron $\mathcal{S}$ and $e = \uparrow, \downarrow$ labeling the two possible configurations of the indicator qubit $\mathcal{E}$. Hence, $\mathbf{U}^{\mathcal{SE}}(t \leftarrow t')$ is a 4×4 matrix that looks like

$$\boxed{\mathbf{U}^{\mathcal{SE}}(t \leftarrow t') = \begin{pmatrix} U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\downarrow} \end{pmatrix},} \tag{22.25}$$

with the functional dependence $(t \leftarrow t')$ suppressed for notational compactness. Equivalently, we can write this matrix as

$$\boxed{\mathbf{U}^{\mathcal{SE}}(t \leftarrow t') = \begin{pmatrix} \begin{pmatrix} U^{\mathcal{S}}_{11} & U^{\mathcal{S}}_{12} \\ U^{\mathcal{S}}_{21} & U^{\mathcal{S}}_{22} \end{pmatrix} U^{\mathcal{E}}_{\uparrow\uparrow} & \begin{pmatrix} U^{\mathcal{S}}_{11} & U^{\mathcal{S}}_{12} \\ U^{\mathcal{S}}_{21} & U^{\mathcal{S}}_{22} \end{pmatrix} U^{\mathcal{E}}_{\uparrow\downarrow} \\ \begin{pmatrix} U^{\mathcal{S}}_{11} & U^{\mathcal{S}}_{12} \\ U^{\mathcal{S}}_{21} & U^{\mathcal{S}}_{22} \end{pmatrix} U^{\mathcal{E}}_{\downarrow\uparrow} & \begin{pmatrix} U^{\mathcal{S}}_{11} & U^{\mathcal{S}}_{12} \\ U^{\mathcal{S}}_{21} & U^{\mathcal{S}}_{22} \end{pmatrix} U^{\mathcal{E}}_{\downarrow\downarrow} \end{pmatrix},} \tag{22.26}$$

with the 2×2 time-evolution operator $\mathbf{U}^{\mathcal{S}}(t \leftarrow t')$ for the electron multiplying each of the four entries of the 2×2 time-evolution operator $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$ for the indicator qubit. (It's remarkable just how complicated things get even with just a pair of two-configuration systems.)

For simplicity, let's suppose that the indicator qubit doesn't evolve after its interaction with the electron. Then from the wall time t' to the detection time t, the indicator qubit's time-evolution operator can be taken to be the 2×2 identity matrix:

$$\boxed{\mathbf{U}^{\mathcal{E}}(t \leftarrow t') = \begin{pmatrix} U^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t') & U^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t') \\ U^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t') & U^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t') \end{pmatrix} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.} \tag{22.27}$$

The state vector $\mathbf{\Psi}(t)$ for the composite system at the detection time is given by

$$\boxed{\mathbf{\Psi}(t) = \mathbf{U}^{\mathcal{SE}}(t \leftarrow t')\mathbf{\Psi}(t').} \tag{22.28}$$

That is,

$$\boxed{\begin{pmatrix} \Psi_{1,\uparrow}(t) \\ \Psi_{2,\uparrow}(t) \\ \Psi_{1,\downarrow}(t) \\ \Psi_{2,\downarrow}(t) \end{pmatrix} = \begin{pmatrix} U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \Psi_{1,\uparrow}(t') \\ \Psi_{2,\uparrow}(t') \\ \Psi_{1,\downarrow}(t') \\ \Psi_{2,\downarrow}(t') \end{pmatrix}.} \tag{22.29}$$

Recalling our expressions (22.21) for the entries of the composite system's state vector at the wall time t',

$$\boxed{\begin{aligned} \Psi_{1,\uparrow}(t') &= \Psi_1(t'), \\ \Psi_{2,\uparrow}(t') &= 0, \\ \Psi_{1,\downarrow}(t') &= 0, \\ \Psi_{2,\downarrow}(t') &= \Psi_2(t'), \end{aligned}}$$

our matrix equation (22.29) simplifies to

$$\boxed{\begin{pmatrix} \Psi_{1,\uparrow}(t) \\ \Psi_{2,\uparrow}(t) \\ \Psi_{1,\downarrow}(t) \\ \Psi_{2,\downarrow}(t) \end{pmatrix} = \begin{pmatrix} U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{11}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{12}U^{\mathcal{E}}_{\downarrow\downarrow} \\ U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{21}U^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{22}U^{\mathcal{E}}_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \Psi_1(t') \\ 0 \\ 0 \\ \Psi_2(t') \end{pmatrix}.} \tag{22.30}$$

Carrying out the matrix multiplication explicitly, we see that the four entries $\Psi_{1,\uparrow}(t)$, $\Psi_{2,\uparrow}(t)$, $\Psi_{1,\downarrow}(t)$ and $\Psi_{2,\downarrow}(t)$ of the composite system's state vector at the detection time $t > t'$ are given by

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')U^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')U^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\
\Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')U^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')U^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\
\Psi_{1,\downarrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')U^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')U^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\
\Psi_{2,\downarrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')U^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')U^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t')\Psi_2(t'),
\end{aligned}$$

where, in particular, the entry $\Psi_{2,\uparrow}(t)$ corresponds to the possibility that the electron eventually floats into the *lower* portion of the apparatus after it passes through the *upper* slit, and the entry $\Psi_{1,\downarrow}(t)$ corresponds to the possibility that the electron eventually floats into the *upper* portion of the apparatus after it passes through the *lower* slit. Using (22.27) for the entries of the indicator qubit's time-evolution operator $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$, the four expressions above become

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')(1)\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')(0)\Psi_2(t'), \\
\Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')(1)\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')(0)\Psi_2(t'), \\
\Psi_{1,\downarrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')(0)\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')(1)\Psi_2(t'), \\
\Psi_{2,\downarrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')(0)\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')(1)\Psi_2(t'),
\end{aligned}$$

which simplify to

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')\Psi_1(t'), \\
\Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')\Psi_1(t'), \\
\Psi_{1,\downarrow}(t) &= U^{\mathcal{S}}_{12}(t \leftarrow t')\Psi_2(t'), \\
\Psi_{2,\downarrow}(t) &= U^{\mathcal{S}}_{22}(t \leftarrow t')\Psi_2(t').
\end{aligned}$$

Under the identification $\mathbf{U}(t \leftarrow t') \equiv \mathbf{U}^{\mathcal{S}}(t \leftarrow t')$, recall from (22.15) that we already gave names to the four distinct terms appearing on the right-hand sides of these equations:

$$\boxed{\begin{aligned}
\Phi_1(t) &\equiv U^{\mathcal{S}}_{11}(t \leftarrow t')\Psi_1(t'), \\
\Phi_2(t) &\equiv U^{\mathcal{S}}_{21}(t \leftarrow t')\Psi_1(t'), \\
\Omega_1(t) &\equiv U^{\mathcal{S}}_{12}(t \leftarrow t')\Psi_2(t'), \\
\Omega_2(t) &\equiv U^{\mathcal{S}}_{22}(t \leftarrow t')\Psi_2(t').
\end{aligned}} \tag{22.31}$$

Hence,

$$\boxed{\begin{aligned}
\Psi_{1,\uparrow}(t) &= \Phi_1(t), \\
\Psi_{2,\uparrow}(t) &= \Phi_2(t), \\
\Psi_{1,\downarrow}(t) &= \Omega_1(t), \\
\Psi_{2,\downarrow}(t) &= \Omega_2(t).
\end{aligned}} \tag{22.32}$$

On comparing these four expressions with (22.16) from the case where the indicator qubit was *absent*,

$$\boxed{\left.\begin{aligned}
\Psi_1(t) &= \Phi_1(t) + \Omega_1(t), \\
\Psi_2(t) &= \Phi_2(t) + \Omega_2(t)
\end{aligned}\right\} \leftarrow \text{[no indicator qubit]},}$$

we see that with the indicator qubit *present*, the sum $\Phi_1(t) + \Omega_1(t)$ and the sum $\Phi_2(t) + \Omega_2(t)$ are now separated out into their individual terms.

Applying the Born rule (17.19) again, we see that the joint probabilities for the electron and the indicator qubit at the detection time $t > t'$ are given by

$$\boxed{\begin{aligned} p_{1,\uparrow}(t) &= |\Psi_{1,\uparrow}(t)|^2 = |\Phi_1(t)|^2, \\ p_{2,\uparrow}(t) &= |\Psi_{2,\uparrow}(t)|^2 = |\Phi_2(t)|^2, \\ p_{1,\downarrow}(t) &= |\Psi_{1,\downarrow}(t)|^2 = |\Omega_1(t)|^2, \\ p_{2,\downarrow}(t) &= |\Psi_{2,\downarrow}(t)|^2 = |\Omega_2(t)|^2. \end{aligned}} \tag{22.33}$$

We can obtain the standalone probabilities $p_1(t)$ and $p_2(t)$ for the electron alone at the detection time $t > t'$ by marginalizing over the configurations $e = \uparrow, \downarrow$ of the indicator qubit:

$$\boxed{\begin{aligned} p_1(t) &= \sum_{e=\uparrow,\downarrow} p_{1,e}(t) \equiv p_{1,\uparrow}(t) + p_{1,\downarrow}(t) = |\Phi_1(t)|^2 + |\Omega_1(t)|^2, \\ p_2(t) &= \sum_{e=\uparrow,\downarrow} p_{2,e}(t) \equiv p_{2,\uparrow}(t) + p_{2,\downarrow}(t) = |\Phi_2(t)|^2 + |\Omega_2(t)|^2. \end{aligned}} \tag{22.34}$$

Notice here that these results feature absolute-value-squares *and then* sums, meaning that there are no cross-terms and therefore no interference. By comparison, the final standalone probabilities (22.17) for the electron *without* the indicator qubit feature sums *and then* absolute-value-squares,

$$\boxed{\left.\begin{aligned} p_1(t) &= |\Phi_1(t) + \Omega_1(t)|^2, \\ p_2(t) &= |\Phi_2(t) + \Omega_2(t)|^2 \end{aligned}\right\} \leftarrow \text{[no indicator qubit]},}$$

which leads to interference.

Sums *inside* of absolute-value-squares are known as *coherent sums*, whereas sums *outside* of absolute-value-squares are called *incoherent sums*. We therefore see that with the indicator qubit present, we've gone from coherent sums to incoherent sums, an important quantum-theoretic phenomenon called *decoherence*.

Let's now suppose that after the indicator qubit interacts with the electron, and without sharing its reading with any other systems, the indicator qubit 'forgets' its reading, and returns back to its initial configuration $\uparrow$. We can implement this time evolution by replacing the unitary time-evolution operator $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$ defined in (22.27) with a non-unitary (indeed, non-invertible) time-evolution operator $\boldsymbol{\Theta}^{\mathcal{E}}(t \leftarrow t')$ defined by

$$\boxed{\boldsymbol{\Theta}^{\mathcal{E}}(t \leftarrow t') = \begin{pmatrix} \Theta^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t') & \Theta^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t') \\ \Theta^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t') & \Theta^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t') \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.} \tag{22.35}$$

Then our matrix equation (22.30) for the composite system's state vector $\boldsymbol{\Psi}(t)$ at the detection time $t > t'$ becomes

$$\boxed{\begin{pmatrix} \Psi_{1,\uparrow}(t) \\ \Psi_{2,\uparrow}(t) \\ \Psi_{1,\downarrow}(t) \\ \Psi_{2,\downarrow}(t) \end{pmatrix} = \begin{pmatrix} U^{\mathcal{S}}_{11}\Theta^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{12}\Theta^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{11}\Theta^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{12}\Theta^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{21}\Theta^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{22}\Theta^{\mathcal{E}}_{\uparrow\uparrow} & U^{\mathcal{S}}_{21}\Theta^{\mathcal{E}}_{\uparrow\downarrow} & U^{\mathcal{S}}_{22}\Theta^{\mathcal{E}}_{\uparrow\downarrow} \\ U^{\mathcal{S}}_{11}\Theta^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{12}\Theta^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{11}\Theta^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{12}\Theta^{\mathcal{E}}_{\downarrow\downarrow} \\ U^{\mathcal{S}}_{21}\Theta^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{22}\Theta^{\mathcal{E}}_{\downarrow\uparrow} & U^{\mathcal{S}}_{21}\Theta^{\mathcal{E}}_{\downarrow\downarrow} & U^{\mathcal{S}}_{22}\Theta^{\mathcal{E}}_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \Psi_1(t') \\ 0 \\ 0 \\ \Psi_2(t') \end{pmatrix},}$$

which gives the four equations

$$\begin{aligned} \Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')\Theta^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')\Theta^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\ \Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')\Theta^{\mathcal{E}}_{\uparrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')\Theta^{\mathcal{E}}_{\uparrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\ \Psi_{1,\downarrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')\Theta^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')\Theta^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t')\Psi_2(t'), \\ \Psi_{2,\downarrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')\Theta^{\mathcal{E}}_{\downarrow\uparrow}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')\Theta^{\mathcal{E}}_{\downarrow\downarrow}(t \leftarrow t')\Psi_2(t'). \end{aligned}$$

Using the explicit form (22.35) of the indicator qubit's new time-evolution operator $\mathbf{\Theta}^{\mathcal{E}}(t)$, these equations become

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')(1)\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')(1)\Psi_2(t'),\\
\Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')(1)\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')(1)\Psi_2(t'),\\
\Psi_{1,\downarrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')(0)\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')(0)\Psi_2(t'),\\
\Psi_{2,\downarrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')(0)\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')(0)\Psi_2(t'),
\end{aligned}$$

which simplify to

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= U^{\mathcal{S}}_{11}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{12}(t \leftarrow t')\Psi_2(t'),\\
\Psi_{2,\uparrow}(t) &= U^{\mathcal{S}}_{21}(t \leftarrow t')\Psi_1(t') + U^{\mathcal{S}}_{22}(t \leftarrow t')\Psi_2(t'),\\
\Psi_{1,\downarrow}(t) &= 0,\\
\Psi_{2,\downarrow}(t) &= 0.
\end{aligned}$$

Recalling the definitions of $\Phi_1(t)$, $\Omega_1(t)$, $\Phi_2(t)$, and $\Omega_2(t)$ once again from (22.15), we end up with

$$\begin{aligned}
\Psi_{1,\uparrow}(t) &= \Phi_1(t) + \Omega_1(t),\\
\Psi_{2,\uparrow}(t) &= \Phi_2(t) + \Omega_2(t),\\
\Psi_{1,\downarrow}(t) &= 0,\\
\Psi_{2,\downarrow}(t) &= 0.
\end{aligned}$$

Invoking the Born rule (17.19) again, the joint probabilities for the electron and the indicator qubit at the detection time $t > t'$ are now

$$\boxed{\begin{aligned}
p_{1,\uparrow}(t) &= |\Psi_{1,\uparrow}(t)|^2 = |\Phi_1(t) + \Omega_1(t)|^2,\\
p_{2,\uparrow}(t) &= |\Psi_{2,\uparrow}(t)|^2 = |\Phi_2(t) + \Omega_2(t)|^2,\\
p_{1,\downarrow}(t) &= |\Psi_{1,\downarrow}(t)|^2 = 0,\\
p_{2,\downarrow}(t) &= |\Psi_{2,\downarrow}(t)|^2 = 0,
\end{aligned}} \tag{22.36}$$

so the standalone probabilities $p_1(t)$ and $p_2(t)$ for the electron alone at the detection time $t > t'$ are

$$\boxed{\begin{aligned}
p_1(t) &= \sum_{e=\uparrow,\downarrow} p_{1,e}(t) \equiv p_{1,\uparrow}(t) + p_{1,\downarrow}(t) = |\Phi_1(t) + \Omega_1(t)|^2,\\
p_2(t) &= \sum_{e=\uparrow,\downarrow} p_{2,e}(t) \equiv p_{2,\uparrow}(t) + p_{2,\downarrow}(t) = |\Phi_2(t) + \Omega_2(t)|^2.
\end{aligned}} \tag{22.37}$$

Notice that these results feature coherent sums again, just like before we introduced the indicator qubit, so the electron's final probability distribution at the detection time $t > t'$ now exhibits interference.

Keep in mind that the 'forgetting' by the indicator qubit here is a very strong form of forgetting, with no traces or records left on any other systems in the larger environment surrounding our double-slit experiment. Such forgetting would be highly impractical for any real-world, macroscopic systems. By contrast, when a human or electronic computer 'forgets' something, there are essentially always residual traces scattered among air molecules and radiated light. As such, when a realistic environment that's more complicated than our highly idealized qubit carries out a reading on a subject system, the loss of interference—that is, the decoherence—is, to an extremely good approximation, irreversible.

Let's generalize the previous calculation, with an eye toward developing the tools that we'll need later to analyze the measurement process. Once more, we'll study the time interval from the wall time $t' > 0$ to the detection time $t > t'$. By assumption, the electron, regarded as the subject system system $\mathcal{S}$, and the indicator qubit, denoted as a highly simplified 'environment' $\mathcal{E}$, form a closed composite system $\mathcal{SE}$ from the wall time t' to the detection time t. Also by assumption, the wall time t' is just after the indicator qubit has interacted with the electron, so the electron $\mathcal{S}$ and the indicator qubit $\mathcal{E}$ are *individually* closed subsystems during the time interval from t' to t.

Because the composite system $\mathcal{SE}$ is closed, there exists a unitary time-evolution operator $\mathbf{U}^{\mathcal{SE}}(t \leftarrow t')$ that transforms Hilbert-space ingredients from t' to t. As a reminder, because each of the two subsystems $\mathcal{S}$ and $\mathcal{E}$ are individually closed during that same time interval from t' to t, they have their own respective time-evolution operators $\mathbf{U}^{\mathcal{S}}(t \leftarrow t')$ and $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$, and the composite system's time-evolution operator factorizes as a tensor product, as in (22.23):

$$\boxed{\mathbf{U}^{\mathcal{SE}}(t \leftarrow t') = \mathbf{U}^{\mathcal{S}}(t \leftarrow t') \otimes \mathbf{U}^{\mathcal{E}}(t \leftarrow t').}$$

More explicitly, in terms of indices, we have (22.24),

$$\boxed{U^{\mathcal{SE}}_{(i,e),(i',e')}(t \leftarrow t') = U^{\mathcal{S}}_{ii'}(t \leftarrow t')U^{\mathcal{E}}_{ee'}(t \leftarrow t'),}$$

where $i = 1, 2$ labels the two possible configurations of the electron $\mathcal{S}$ and $e = \uparrow, \downarrow$ labels the two possible configurations of the indicator qubit $\mathcal{E}$. Rather than specify a particular time-evolution operator $\mathbf{U}^{\mathcal{E}}(t \leftarrow t')$ for the indicator qubit, we'll now allow it to be more general.

We'll also assume that at the initial time $t = 0$, the electron $\mathcal{S}$ is in its jth configuration (say, $j = 1$) and the indicator qubit $\mathcal{E}$ is in its $e = \uparrow$ configuration, so that the composite system $\mathcal{SE}$ has a 4×1 state vector $\mathbf{\Psi}(t)$ at least from $t = 0$ to the detection time $t > t'$. At the wall time t', just after the indicator qubit has read the configuration of the electron, we recall from (22.21) that the composite system's state vector $\mathbf{\Psi}(t')$ has entries

$$\boxed{\begin{aligned} &\Psi_{1,\uparrow}(t') = \Psi_1(t'), \\ &\Psi_{2,\uparrow}(t') = 0, \\ &\Psi_{1,\downarrow}(t') = 0, \\ &\Psi_{2,\downarrow}(t') = \Psi_2(t'). \end{aligned}}$$

Letting

$$\boxed{e(1) \equiv \uparrow, \quad e(2) \equiv \downarrow,} \tag{22.38}$$

we can write the entries of the composite system's state vector $\mathbf{\Psi}(t)$ at the wall time t' more succinctly as

$$\Psi_{i',e'}(t') = \begin{cases} \Psi_{i'}(t') & \text{for } e' = e(i'), \\ 0 & \text{for } e' \neq e(i'), \end{cases}$$

or, even more succinctly, as

$$\boxed{\Psi_{i',e'}(t') = \Psi_{i'}(t')\delta_{e'e(i')},} \tag{22.39}$$

where we use primes on our labels i' and e' just to emphasize that they correspond to the wall time t', and where we've introduced the Kronecker delta

$$\boxed{\delta_{e'e(i')} \equiv \begin{cases} 1 & \text{for } e' = e(i'), \\ 0 & \text{for } e' \neq e(i'). \end{cases}} \tag{22.40}$$

As in (22.28), the composite system's state vector $\mathbf{\Psi}(t)$ at the detection time $t > t'$ is given by acting with the compoite system's time-evolution operator $\mathbf{U}^{\mathcal{SE}}(t \leftarrow t')$ on the composite system's state vector $\mathbf{\Psi}(t')$ at the wall time t':

$$\boxed{\mathbf{\Psi}(t) = \mathbf{U}^{\mathcal{SE}}(t \leftarrow t')\mathbf{\Psi}(t').}$$

Working in terms of individual entries, and using the usual rules of matrix multiplication, this equation becomes

$$\Psi_{i,e}(t) = \sum_{i'=1,2} \sum_{e'=\uparrow,\downarrow} U^{\mathcal{SE}}_{(i,e),(i',e')}(t \leftarrow t')\Psi_{i',e'}(t').$$

Suppressing the argument $(t \leftarrow t')$ to keep the notation as simple as possible, we can write this equation as

$$\Psi_{i,e}(t) = \sum_{i'} \sum_{e'} U^{\mathcal{SE}}_{(i,e),(i',e')}\Psi_{i',e'}(t').$$

Invoking the tensor-factorization (22.24) of the time-evolution operator $\mathbf{U}^{\mathcal{SE}}(t \leftarrow t')$, together with the compact expression (22.39) for the composite system's state vector $\mathbf{\Psi}(t')$ at the wall time t', we have

$$\begin{aligned}\Psi_{i,e}(t) &= \sum_{i'} \sum_{e'} \left[U^{\mathcal{S}}_{ii'} U^{\mathcal{E}}_{ee'}\right] \left[\Psi_{i'}(t')\delta_{e'e(i')}\right] \\ &= \sum_{i'} U^{\mathcal{S}}_{ii'} \Psi_{i'}(t') \left[\sum_{e'} U^{\mathcal{E}}_{ee'} \delta_{e'e(i')}\right].\end{aligned}$$

A summation involving a Kronecker delta collapses to a single term, and, in particular, the summation in brackets on $e' = \uparrow, \downarrow$ collapses to the single term $U^{\mathcal{E}}_{ee(i')}$. To see why, notice that the summation in brackets is

$$\begin{aligned}\sum_{e'} U^{\mathcal{E}}_{ee'} \delta_{e'e(i')} &\equiv U^{\mathcal{E}}_{e\uparrow} \delta_{\uparrow e(i')} + U^{\mathcal{E}}_{e\downarrow} \delta_{\downarrow e(i')} \\ &= \begin{cases} U^{\mathcal{E}}_{e\uparrow} + 0 & \text{for } e(i') = \uparrow, \\ 0 + U^{\mathcal{E}}_{e\downarrow} & \text{for } e(i') = \downarrow \end{cases} \\ &= U^{\mathcal{E}}_{ee(i')},\end{aligned}$$

so our expression for $\Psi_{i,e}(t)$ simplifies to

$$\boxed{\Psi_{i,e}(t) = \sum_{i'} U^{\mathcal{S}}_{ii'} \Psi_{i'}(t') U^{\mathcal{E}}_{ee(i')}.} \tag{22.41}$$

From the Born rule (17.19), the probability of the composite system $\mathcal{SE}$ being in its configuration (i, e) at the detection time $t > t'$ is given by

$$p_{i,e}(t) = |\Psi_{i,e}(t)|^2 = \overline{\Psi_{i,e}(t)} \Psi_{i,e}(t).$$

Inserting our expression (22.41) for $\Psi_{i,e}(t)$ gives

$$p_{i,e}(t) = \overline{\sum_{i'_1} U^{\mathcal{S}}_{ii'_1} \Psi_{i'_1}(t') U^{\mathcal{E}}_{ee(i'_1)}} \sum_{i'_2} U^{\mathcal{S}}_{ii'_2} \Psi_{i'_2}(t') U^{\mathcal{E}}_{ee(i'_2)},$$

where we've relabeled the index of the first summation from i' to i'_1 and relabeled the index of the second summation from i' to i'_2 to make clear that these are two distinct summations. Rearranging, we arrive at the complicated-looking expression

$$\boxed{p_{i,e}(t) = \sum_{i'_1} \sum_{i'_2} \overline{U^{\mathcal{S}}_{ii'_1}} U^{\mathcal{S}}_{ii'_2} \overline{\Psi_{i'_1}(t')} \Psi_{i'_2}(t') \overline{U^{\mathcal{E}}_{ee(i'_1)}} U^{\mathcal{E}}_{ee(i'_2)}.} \tag{22.42}$$

To obtain the standalone probability $p_i(t)$ for the electron to be in its ith configuration at the detection time $t > t'$, we marginalize over the indicator qubit's two configurations $e = \uparrow, \downarrow$, as usual:

$$p_i(t) = \sum_{e=\uparrow,\downarrow} p_{i,e}(t) \equiv p_{i,\uparrow}(t) + p_{i,\downarrow}(t).$$

Substituting in our expression (22.42) for the composite system's probability $p_{i,e}(t)$ gives a formula that, at first glance, looks even more complicated than before:

$$p_i(t) = \sum_{e} \sum_{i'_1} \sum_{i'_2} \overline{U^{\mathcal{S}}_{ii'_1}} U^{\mathcal{S}}_{ii'_2} \overline{\Psi_{i'_1}(t')} \Psi_{i'_2}(t') \overline{U^{\mathcal{E}}_{ee(i'_1)}} U^{\mathcal{E}}_{ee(i'_2)}.$$

Rearranging the summations, we have

$$p_i(t) = \sum_{i'_1} \sum_{i'_2} \overline{U^{\mathcal{S}}_{ii'_1}} U^{\mathcal{S}}_{ii'_2} \overline{\Psi_{i'_1}(t')} \Psi_{i'_2}(t') \left[\sum_{e} \overline{U^{\mathcal{E}}_{ee(i'_1)}} U^{\mathcal{E}}_{ee(i'_2)}\right]. \tag{22.43}$$

For now, let's look at just the summation on $e = \uparrow, \downarrow$ in brackets:

$$\sum_e \overline{U^{\mathcal{E}}_{ee(i_1')}} U^{\mathcal{E}}_{ee(i_2')}.$$

Recalling that the adjoint operation $\dagger$ is defined for any matrix $\mathbf{X}$ by transposition and complex-conjugation, $\left(X^\dagger\right)_{ij} \equiv \overline{X_{ji}}$, we have $\overline{U^{\mathcal{E}}_{ee(i_1')}} = U^{\mathcal{E}\dagger}_{e(i_1')e}$, so we can write the expression above as

$$\sum_e U^{\mathcal{E}\dagger}_{e(i_1')e} U^{\mathcal{E}}_{ee(i_2')}.$$

In this form, we see that we're multiplying the matrix $\mathbf{U}^{\mathcal{E}\dagger}$ and the matrix $\mathbf{U}^{\mathcal{E}}$, and because these are unitary matrices, the result is just the 2×2 identity matrix $\mathbf{1}$:

$$\mathbf{U}^{\mathcal{E}\dagger}\mathbf{U}^{\mathcal{E}} = \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The entries of the identity matrix are given by the Kronecker delta, so we conclude that

$$\sum_e U^{\mathcal{E}\dagger}_{e(i_1')e} U^{\mathcal{E}}_{ee(i_2')} = \delta_{i_1' i_2'} \equiv \begin{cases} 1 & \text{for } i_1' = i_2', \\ 0 & \text{for } i_1' \neq i_2'. \end{cases}$$

Inserting this result into our formula (22.43) for the electron's standalone probability $p_i(t)$ at the detection time $t > t'$, we obtain

$$p_i(t) = \sum_{i_1'} \sum_{i_2'} \overline{U^{\mathcal{S}}_{ii_1'}} U^{\mathcal{S}}_{ii_2'} \overline{\Psi_{i_1'}(t')} \Psi_{i_2'}(t') \left[\delta_{i_1' i_2'}\right].$$

The Kronecker delta $\delta_{i_1' i_2'}$ is equal to 1 if $i_2' = i_1'$ and is equal to 0 if $i_2' \neq i_1'$, so the summation on i_2' collapses to the single term for which $i_2' = i_1'$, thereby giving us

$$p_i(t) = \sum_{i_1'} \overline{U^{\mathcal{S}}_{ii_1'}} U^{\mathcal{S}}_{ii_1'} \overline{\Psi_{i_1'}(t')} \Psi_{i_1'}(t').$$

Now that we're down to just a single summation, we're free to relabel the summation index i_1' back to i' to make the notation simpler:

$$p_i(t) = \sum_{i'} \overline{U^{\mathcal{S}}_{ii'}} U^{\mathcal{S}}_{ii'} \overline{\Psi_{i'}(t')} \Psi_{i'}(t').$$

Using $\overline{z}z = |z|^2$ again to write

$$\overline{U^{\mathcal{S}}_{ii'}} U^{\mathcal{S}}_{ii'} = |U^{\mathcal{S}}_{ii'}|^2, \qquad \overline{\Psi_{i'}(t')} \Psi_{i'}(t') = |\Psi_{i'}(t')|^2,$$

and restoring the argument $(t \leftarrow t')$, we obtain

$$\boxed{p_i(t) = \sum_{i'=1,2} |U^{\mathcal{S}}_{ii'}(t \leftarrow t')|^2 |\Psi_{i'}(t')|^2.} \tag{22.44}$$

From the Born rule, we know that $p_{i'}(t') = |\Psi_{i'}(t')|^2$ is just the probability that the electron is in its i'th configuration at the wall time t', and recalling (22.1), we know that $|U^{\mathcal{S}}_{ii'}(t \leftarrow t')|^2$ defines the entries of a unistochastic matrix $\mathbf{\Gamma}^{\mathcal{S}}(t \leftarrow t')$,

$$\boxed{\Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') \equiv |U^{\mathcal{S}}_{ii'}(t \leftarrow t')|^2.} \tag{22.45}$$

We can therefore recast our expression (22.44) for the electron's standalone probability $p_i(t)$ at the detection time $t > t'$ more succinctly as

$$\boxed{p_i(t) = \sum_{i'=1,2} \Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') p_{i'}(t').} \tag{22.46}$$

This equation has precisely the form of a linear marginalization relationship (9.1) relating arbitrary standalone probabilities $p_{i'}(t')$ for the electron at the time t' to the corresponding standalone probabilities $p_i(t)$ at the time t. This marginalization relationship looks just like our original marginalization formula (16.15), but with t' playing the role of a new '$t = 0$.'

Applying our original marginalization formula (16.15) to obtain the electron's standalone probabilities $p_{i'}(t')$ at the wall time t' gives

$$\boxed{p_{i'}(t') = \sum_{j=1,2} \Gamma^{\mathcal{S}}_{i'j}(t')p_j(0).} \tag{22.47}$$

Inserting this formula into the right-hand side of (22.46) yields

$$p_i(t) = \sum_{i'=1,2} \Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') \left[\sum_{j=1,2} \Gamma^{\mathcal{S}}_{i'j}(t')p_j(0) \right].$$

Rearranging, we find

$$p_i(t) = \sum_{j=1,2} \left[\sum_{i'=1,2} \Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t')\Gamma^{\mathcal{S}}_{i'j}(t') \right] p_j(0).$$

This formula is precisely a linear marginalization relationship between the electron's standalone probabilities $p_j(0)$ at the initial time $t = 0$ and its standalone probabilities $p_i(t)$ at the detection time t:

$$\boxed{p_i(t) = \sum_{j=1,2} \Gamma^{\mathcal{S}}_{ij}(t)p_j(0),} \tag{22.48}$$

with an overall stochastic matrix $\mathbf{\Gamma}^{\mathcal{S}}(t)$ given by

$$\boxed{\Gamma^{\mathcal{S}}_{ij}(t) \equiv \sum_{i'=1,2} \Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t')\Gamma^{\mathcal{S}}_{i'j}(t'),} \tag{22.49}$$

or, equivalently,

$$\boxed{\mathbf{\Gamma}^{\mathcal{S}}(t) \equiv \mathbf{\Gamma}^{\mathcal{S}}(t \leftarrow t')\mathbf{\Gamma}^{\mathcal{S}}(t').} \tag{22.50}$$

The relation (22.50) is precisely of the form (15.1) for a stochastic matrix that's *divisible* at t', so it would be natural to call t' a *division event*. What we learn from this calculation is that the interaction at the wall time t' between the electron and the indicator qubit, in which the indicator qubit reads the electron's configuration, corresponds to a division event for the electron's stochastic dynamics. As a corollary, we see that $t = 0$ is not special in being a division event, but that division events for systems will occur whenever an 'environment' of some kind reads the system's configuration.

We can learn even more from the foregoing calculation. Notice that if the electron's probability distribution $p_{i'}(t')$ at the wall time t' happens to be pure, so that $p_{i'}(t') = 1$ for a specific value of the configuration i' and is equal to 0 for all other configurations, then the marginalization formula (22.46) reduces to

$$p_i(t) = \Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') \leftarrow [\text{if the electron is definitely in its configuration } i' \text{ at } t'].$$

This formula tells us immediately that $\Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t')$, as we might have expected, is the conditional probability for the electron to be in its ith configuration at t, given that it's in its i'th configuration at t':

$$\boxed{\Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') = p(i, t|i', t').} \tag{22.51}$$

Turning this equation around and invoking the definition $\Gamma^{\mathcal{S}}_{ii'}(t \leftarrow t') \equiv |U^{\mathcal{S}}_{ii'}(t \leftarrow t')|^2$ from (22.45), we have

$$p(i, t|i', t') = |U^{\mathcal{S}}_{ii'}(t \leftarrow t')|^2.$$

Re-expressing the right-hand side in dictionary form, akin to (16.9), gives

$$p(i,t|i',t') = \mathrm{tr}(\mathbf{U}^{\mathcal{S}\dagger}(t \leftarrow t')\mathbf{P}_i^{\mathcal{S}}\mathbf{U}^{\mathcal{S}}(t \leftarrow t')\mathbf{P}_{i'}^{\mathcal{S}}).$$

Using the cyclic property of the trace, (11.27), we can move the matrix $\mathbf{U}^\dagger(t \leftarrow t')$ over to the other side of the trace:

$$\boxed{p(i,t|i',t') = \mathrm{tr}\big(\mathbf{P}_i^{\mathcal{S}}\left[\mathbf{U}^{\mathcal{S}}(t \leftarrow t')\mathbf{P}_{i'}^{\mathcal{S}}\mathbf{U}^{\mathcal{S}\dagger}(t \leftarrow t')\right]\big).} \tag{22.52}$$

Comparing this expression with the formula (16.23) for the standalone probability $p_i(t)$ in terms of a trace over the product of the ith configuration projector $\mathbf{P}_i^{\mathcal{S}}$ and the electron's density matrix $\boldsymbol{\rho}^{\mathcal{S}}(t)$,

$$\boxed{p_i(t) = \mathrm{tr}\big(\mathbf{P}_i^{\mathcal{S}}\boldsymbol{\rho}^{\mathcal{S}}(t)\big) = \mathrm{tr}\big(\mathbf{P}_i^{\mathcal{S}}\left[\mathbf{U}^{\mathcal{S}}(t)\boldsymbol{\rho}^{\mathcal{S}}(0)\mathbf{U}^{\mathcal{S}\dagger}(t)\right]\big),}$$

we see immediately that in conditioning on the electron's configuration i' at the wall time t'—or, equivalently, conditioning on the indicator qubit's *reading* at t'—we have effectively replaced the electron's density matrix $\boldsymbol{\rho}^{\mathcal{S}}(t)$ with

$$\mathbf{U}^{\mathcal{S}}(t \leftarrow t')\mathbf{P}_{i'}^{\mathcal{S}}\mathbf{U}^{\mathcal{S}\dagger}(t \leftarrow t'),$$

which reduces at the wall time t' itself to just the single projector

$$\mathbf{P}_{i'}^{\mathcal{S}} = \mathbf{e}_{i'}\mathbf{e}_{i'}^\dagger.$$

That is, conditioning on i' at t' is equivalent to *collapsing* the electron's density matrix at t' to $\mathbf{P}_{i'}^{\mathcal{S}}$. In other words, after conditioning on i' at t', we should use t' in place of $t = 0$, we should use $\mathbf{U}^{\mathcal{S}}(t \leftarrow t')$ in place of $\mathbf{U}^{\mathcal{S}}(t)$, and we should use $\mathbf{P}_{i'}^{\mathcal{S}}$ in place of $\boldsymbol{\rho}^{\mathcal{S}}(0)$. Hence, whatever the electron's state vector or wave function $\boldsymbol{\Psi}^{\mathcal{S}}(t' - \epsilon)$ was at a small time ϵ *just before* t'—that is, just before the interaction with the indicator qubit—we should replace that state vector with $\mathbf{e}_{i'}$ starting at t':

$$\boxed{\boldsymbol{\Psi}^{\mathcal{S}}(t' - \epsilon) \mapsto \mathbf{e}_{i'}.} \tag{22.53}$$

This formula represents the simplest version of *wave-function collapse*, which we see arises from a special case of conditioning.

Measurement in the Double-Slit Experiment

To clarify what's going on here, let's now study the time interval from the initial $t = 0$ to the wall time t', just after the indicator qubit in our double-slit experiment has interacted with the electron. This time interval will capture the simplest kind of measurement in quantum theory, and will serve as the basic paradigm for measurements of quantum-theoretic observables more generally—even beyond the special case of random variables, which are represented by diagonal matrices.

By assumption, the initial configuration of the composite system $\mathcal{SE}$ consisting of the electron $\mathcal{S}$ and the indicator qubit $\mathcal{E}$ at $t = 0$ is $(j, \uparrow)$ for some specific value of the electron's configuration label j—say, $j = 1$. The composite system therefore has an initial state vector that we can take to be

$$\boxed{\boldsymbol{\Psi}(0) = \begin{pmatrix} \Psi_{(1,\uparrow)}(0) \\ \Psi_{(2,\uparrow)}(0) \\ \Psi_{(1,\downarrow)}(0) \\ \Psi_{(2,\downarrow)}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},} \tag{22.54}$$

or, equivalently,

$$\boxed{\Psi_{j,e_0}(0) = \delta_{j1}\delta_{e_0\uparrow} = \begin{cases} 1 & \text{for } (j, e_0) = (1, \uparrow), \\ 0 & \text{for } (j, e_0) \neq (1, \uparrow), \end{cases}} \tag{22.55}$$

where we use the special label e_0 here to emphasize that this index refers to the indicator qubit's configuration at the initial time $t = 0$.

This state vector is simple enough that it admits a special kind of factorization. To start, let's introduce two 2×1 state vectors $\mathbf{\Psi}^{\mathcal{S}}(0)$ and $\mathbf{\Psi}^{\mathcal{E}}(0)$ that are respectively defined to be the configuration-basis vectors $\mathbf{e}_1$ and $\mathbf{e}_\uparrow$, which we can write as mathematically identical column vectors:

$$\boxed{\begin{aligned} \mathbf{\Psi}^{\mathcal{S}}(0) &\equiv \begin{pmatrix} \Psi_1^{\mathcal{S}}(0) \\ \Psi_2^{\mathcal{S}}(0) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1, \\ \mathbf{\Psi}^{\mathcal{E}}(0) &\equiv \begin{pmatrix} \Psi_\uparrow^{\mathcal{E}}(0) \\ \Psi_\downarrow^{\mathcal{E}}(0) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_\uparrow. \end{aligned}} \tag{22.56}$$

That is,

$$\boxed{\begin{aligned} \Psi_j^{\mathcal{S}}(0) &= \delta_{j1} = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j = 2, \end{cases} \\ \Psi_{e_0}^{\mathcal{E}}(0) &= \delta_{e_0\uparrow} = \begin{cases} 1 & \text{for } e_0 = \uparrow, \\ 0 & \text{for } e_0 = \downarrow. \end{cases} \end{aligned}} \tag{22.57}$$

Notice then from (22.54) that

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Psi_1^{\mathcal{S}}(0)\Psi_\uparrow^{\mathcal{E}}(0) \\ \Psi_2^{\mathcal{S}}(0)\Psi_\uparrow^{\mathcal{E}}(0) \\ \Psi_1^{\mathcal{S}}(0)\Psi_\downarrow^{\mathcal{E}}(0) \\ \Psi_2^{\mathcal{S}}(0)\Psi_\downarrow^{\mathcal{E}}(0) \end{pmatrix}, \tag{22.58}$$

or, equivalently, from (22.55), that

$$\Psi_{j,e_0}(0) = \delta_{j1}\delta_{e_0\uparrow} = \Psi_j^{\mathcal{S}}(0)\Psi_{e_0}^{\mathcal{E}}(0). \tag{22.59}$$

We can capture this factorization by writing the composite system's state vector as the *tensor product*

$$\boxed{\mathbf{\Psi}(0) = \mathbf{e}_1 \otimes \mathbf{e}_\uparrow.} \tag{22.60}$$

If we assume that the indicator qubit remains unchanging until the interaction with the electron just before the wall time t', then for a small time interval ϵ, we can model the time evolution of the composite system up until that interaction as the unitary 4×4 matrix

$$\boxed{\mathbf{U}^{\mathcal{S}\mathcal{E}}(t' - \epsilon) \equiv \mathbf{U}^{\mathcal{S}}(t' - \epsilon) \otimes \mathbf{1}.} \tag{22.61}$$

Here $\mathbf{U}^{\mathcal{S}}(t' - \epsilon)$ handles the time evolution of the electron from $t = 0$ to $t' - \epsilon$, whereas the time evolution of the indicator qubit is trivial during that time interval, as is captured by the identity matrix $\mathbf{1}$. The interaction takes place from $t' - \epsilon$ to t', and a simple way to model it is using the unitary 4×4 matrix defined by

$$\boxed{\mathbf{U}^{\mathcal{S}\mathcal{E}}(t' \leftarrow t' - \epsilon) \equiv \sum_{i'=1}^{2} \mathbf{P}_{i'}^{\mathcal{S}} \otimes \mathbf{R}_{e(i')}^{\mathcal{E}} \equiv \mathbf{P}_1^{\mathcal{S}} \otimes \mathbf{R}_\uparrow^{\mathcal{E}} + \mathbf{P}_2^{\mathcal{S}} \otimes \mathbf{R}_\downarrow^{\mathcal{E}},} \tag{22.62}$$

where $\mathbf{R}_\uparrow^{\mathcal{E}}$ is a 2×2 unitary matrix that leaves $\mathbf{e}_\uparrow$ unchanged and $\mathbf{R}_\downarrow^{\mathcal{E}}$ is a 2×2 unitary matrix that transforms $\mathbf{e}_\uparrow$ to $\mathbf{e}_\downarrow$:

$$\boxed{\mathbf{R}_\uparrow^{\mathcal{E}}\mathbf{e}_\uparrow = \mathbf{e}_\uparrow, \qquad \mathbf{R}_\downarrow^{\mathcal{E}}\mathbf{e}_\uparrow = \mathbf{e}_\downarrow.} \tag{22.63}$$

For example, we could take

$$\boxed{\mathbf{R}_\uparrow^{\mathcal{E}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \mathbf{1}, \quad \mathbf{R}_\downarrow^{\mathcal{E}} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},} \tag{22.64}$$

where, by coincidence, $\mathbf{R}_\downarrow^{\mathcal{E}}$ happens to have precisely the same form as the first Pauli sigma matrix σ_x, as defined in (12.58). Altogether, then, the time-evolution operator $\mathbf{U}^{\mathcal{S}\mathcal{E}}(t')$ for the composite system from 0 to t' is given by

$$\boxed{\mathbf{U}^{\mathcal{S}\mathcal{E}}(t') \equiv \mathbf{U}^{\mathcal{S}\mathcal{E}}(t' \leftarrow t' - \epsilon)\mathbf{U}^{\mathcal{S}\mathcal{E}}(t' - \epsilon).} \tag{22.65}$$

Remember that all we're doing by introducing this time-evolution operator is defining the entries of a corresponding unistochastic matrix $\mathbf{\Gamma}^{\mathcal{S}\mathcal{E}}(t')$, as usual, where this unistochastic matrix captures how the composite system $\mathcal{S}\mathcal{E}$ probabilistically evolves with time.

We're now ready to calculate the composite system's state vector $\mathbf{\Psi}(t')$ at the wall time t', just after the measurement interaction between the electron $\mathcal{S}$ and the indicator qubit $\mathcal{E}$. Acting with the time-evolution operator $\mathbf{U}^{\mathcal{S}\mathcal{E}}(t')$ on the composite system's initial state vector $\mathbf{\Psi}(0)$, in accordance with the general rule (17.12), we find

$$\begin{aligned}\mathbf{\Psi}(t') &= \mathbf{U}^{\mathcal{S}\mathcal{E}}(t')\mathbf{\Psi}(0)\\ &= \mathbf{U}^{\mathcal{S}\mathcal{E}}(t' \leftarrow t'-\epsilon)\mathbf{U}^{\mathcal{S}\mathcal{E}}(t'-\epsilon)\mathbf{\Psi}(0)\\ &= \left[\sum_{i'=1}^{2}\mathbf{P}_{i'}^{\mathcal{S}} \otimes \mathbf{R}_{e(i')}^{\mathcal{E}}\right]\left[\mathbf{U}^{\mathcal{S}}(t'-\epsilon)\otimes \mathbf{1}\right]\left[\mathbf{e}_1 \otimes \mathbf{e}_{\uparrow}\right].\end{aligned}$$

It is in the nature of tensor products that they sequester indices on matrices in precisely such a way that

$$\boxed{(\mathbf{X}\otimes\mathbf{Y})(\mathbf{v}\otimes\mathbf{w}) = (\mathbf{X}\mathbf{v})\otimes(\mathbf{Y}\mathbf{w}).} \tag{22.66}$$

Hence,

$$\mathbf{\Psi}(t') = \sum_{i'=1}^{2}\left[\mathbf{P}_{i'}^{\mathcal{S}}\mathbf{U}^{\mathcal{S}}(t'-\epsilon)\mathbf{e}_1\right]\otimes\left[\mathbf{R}_{e(i')}^{\mathcal{E}}\mathbf{e}_{\uparrow}\right].$$

By definition, acting with $\mathbf{U}^{\mathcal{S}}(t'-\epsilon)$ on $\mathbf{\Psi}^{\mathcal{S}}(0) \equiv \mathbf{e}_1$ gives the electron's state vector $\mathbf{\Psi}^{\mathcal{S}}(t'-\epsilon)$ at the time $t'-\epsilon$ just before the measurement interaction with the indicator qubit,

$$\mathbf{U}^{\mathcal{S}}(t'-\epsilon)\mathbf{\Psi}^{\mathcal{S}}(0) = \mathbf{\Psi}^{\mathcal{S}}(t'-\epsilon),$$

and acting on the result with the projector $\mathbf{P}_{i'}^{\mathcal{S}}$ yields

$$\begin{aligned}\mathbf{P}_{i'}^{\mathcal{S}}\mathbf{\Psi}^{\mathcal{S}}(t'-\epsilon) = \mathbf{P}_{i'}^{\mathcal{S}}\begin{pmatrix}\Psi_1^{\mathcal{S}}(t'-\epsilon)\\ \Psi_2^{\mathcal{S}}(t'-\epsilon)\end{pmatrix} &= \begin{cases}\begin{pmatrix}\Psi_1^{\mathcal{S}}(t'-\epsilon)\\ 0\end{pmatrix} & \text{for } i'=1,\\[2ex] \begin{pmatrix}0\\ \Psi_2^{\mathcal{S}}(t'-\epsilon)\end{pmatrix} & \text{for } i'=2\end{cases}\\ &= \Psi_{i'}^{\mathcal{S}}(t'-\epsilon)\mathbf{e}_{i'}.\end{aligned}$$

Meanwhile, we know from (22.63) that

$$\mathbf{R}_{e(i')}^{\mathcal{E}}\mathbf{e}_{\uparrow} = \mathbf{e}_{e(i')}.$$

Hence,

$$\begin{aligned}\mathbf{\Psi}(t') &= \sum_{i'=1}^{2}\left[\Psi_{i'}^{\mathcal{S}}(t'-\epsilon)\mathbf{e}_{i'}\right]\otimes\left[\mathbf{e}_{e(i')}\right]\\ &= \sum_{i'=1}^{2}\Psi_{i'}^{\mathcal{S}}(t'-\epsilon)\left[\mathbf{e}_{i'}\otimes\mathbf{e}_{e(i')}\right].\end{aligned}$$

Taking the limit $\epsilon \to 0$ of a very short measurement interaction, we therefore end up with

$$\begin{aligned}
\boldsymbol{\Psi}(t') &= \sum_{i'=1}^{2} \Psi_{i'}^{\mathcal{S}}(t') \big[\mathbf{e}_{i'} \otimes \mathbf{e}_{e(i')}\big] \\
&= \Psi_1^{\mathcal{S}}(t')[\mathbf{e}_1 \otimes \mathbf{e}_\uparrow] + \Psi_2^{\mathcal{S}}(t')[\mathbf{e}_2 \otimes \mathbf{e}_\downarrow] \\
&= \Psi_1^{\mathcal{S}}(t') \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \Psi_2^{\mathcal{S}}(t') \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \Psi_1^{\mathcal{S}}(t') \\ 0 \\ 0 \\ \Psi_2^{\mathcal{S}}(t') \end{pmatrix}.
\end{aligned}$$

The result is therefore precisely the required form (22.39) of the state vector for the composite system at the wall time t'.

At the end of this interaction, at t', we know from (22.22) that the composite system $\mathcal{SE}$ has joint probabilities

$$\begin{aligned}
p_{1,\uparrow}(t') &= |\Psi_{1,\uparrow}(t')|^2 = |\Psi_1^{\mathcal{S}}(t')|^2, \\
p_{2,\uparrow}(t') &= |\Psi_{2,\uparrow}(t')|^2 = 0, \\
p_{1,\downarrow}(t') &= |\Psi_{1,\downarrow}(t')|^2 = 0, \\
p_{2,\downarrow}(t') &= |\Psi_{2,\downarrow}(t')|^2 = |\Psi_2^{\mathcal{S}}(t')|^2.
\end{aligned}$$

Hence, if we marginalize over the electron's configurations $i = 1, 2$, then we see that the indicator qubit's standalone probabilities at t' are

$$\boxed{\begin{aligned}
p_\uparrow(t') &= p_{1,\uparrow}(t') + p_{2,\uparrow}(t') = |\Psi_1^{\mathcal{S}}(t')|^2, \\
p_\downarrow(t') &= p_{1,\downarrow}(t') + p_{2,\downarrow}(t') = |\Psi_2^{\mathcal{S}}(t')|^2.
\end{aligned}} \tag{22.67}$$

These equations mean that $|\Psi_{i'}^{\mathcal{S}}(t')|^2$ not only tells us the probability of the *electron* being in its i'th configuration at t', but also the probability of the *indicator qubit* being in its $e(i')$th configuration at t':

$$\boxed{p_{e(i')}(t') = |\Psi_{i'}^{\mathcal{S}}(t')|^2.} \tag{22.68}$$

This result is another, more general form of the *Born rule*—the Born rule for *measurement outcomes* or *readings* on measuring devices. That is, $|\Psi_{i'}^{\mathcal{S}}(t')|^2$ also has the meaning of the standalone probability for the measuring device in this case—namely, the indicator qubit—to be in an outcome-configuration $e(i')$ corresponding to the appropriate value of the electron's configuration label i'.