A Dataset & Signal Analysis Interpretation of Quantum Mechanics

Peter Morgan

Yale University
peter.w.morgan@yale.edu

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Bhadrak Autonomous College, Odisha (by Meet)

Thank you to Rajat Kumar Pradhan for
inviting me to be part of India's National Science Day,
marking the discovery of the Raman effect by Sir CV Raman

YouTube Video available at https://youtu.be/VnDD_MfVbeI

Part I: From the Data to Quantum Mechanics

- Quantum Mechanics and Classical Mechanics — as *Dataset* formalisms NOT about ensembles of particles #5
- Classical Mechanics ⟶ CM_+, with (1) *more Poisson* and with (2) quantum noise #7
  - The Measurement Problem ⟵ noncommutativity #12
- Transition to Part II: Bell inequalities for Noisy classical fields #16

Part II: Quantum Field Theory as Signal Analysis

- A first look at Quantum Field Theory ⟵ with signal analysis in mind #21
  - Noisy classical fields, QNDFT, constructed within QFT Tsang&Caves PRX 2012 QM–Free-Subsystems #25

An evolution of ideas in:

"Classical states, quantum field measurement", Physica Scripta 2019
"An algebraic approach to Koopman classical mechanics", Annals of Physics 2020
"The collapse of a quantum state as a joint probability construction", Journal of Physics A 2022

and, ancient history, "Bell inequalities for random fields", Journal of Physics A 2006

a Dataset&Signal Analysis Interpretation of QM — outline

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Part III: (not today)
arXiv:2109.04412

- Nonlinear axioms for QFT&QNDFT ⟵ signal analysis and renormalization
- Towards Quantum&QND Gravity ⟵ a measurement theoretic path

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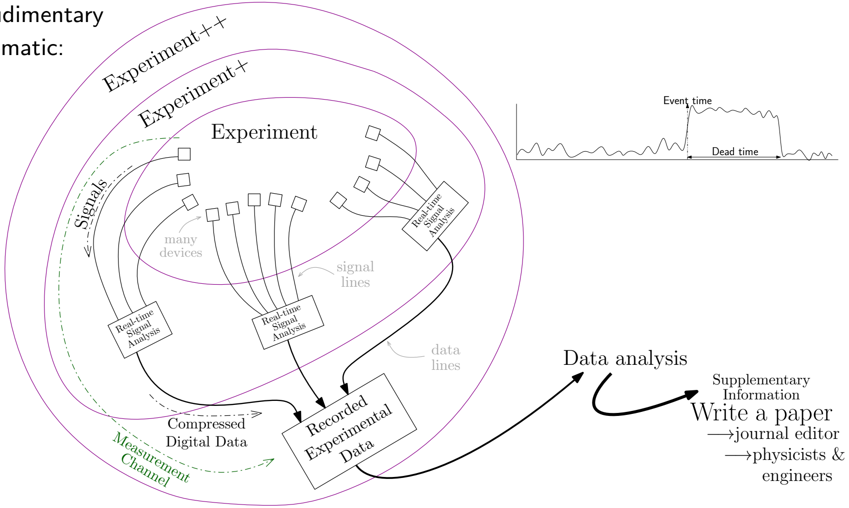
⟵ #1 ⟶

A rudimentary schematic:



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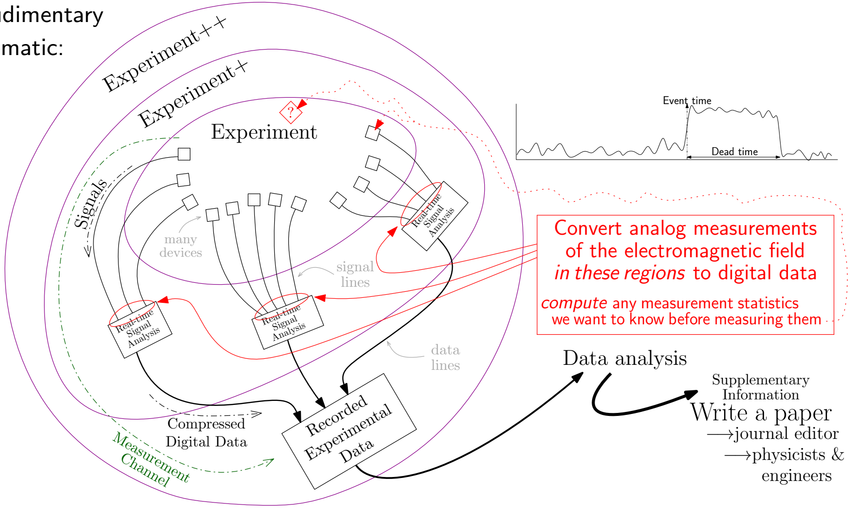

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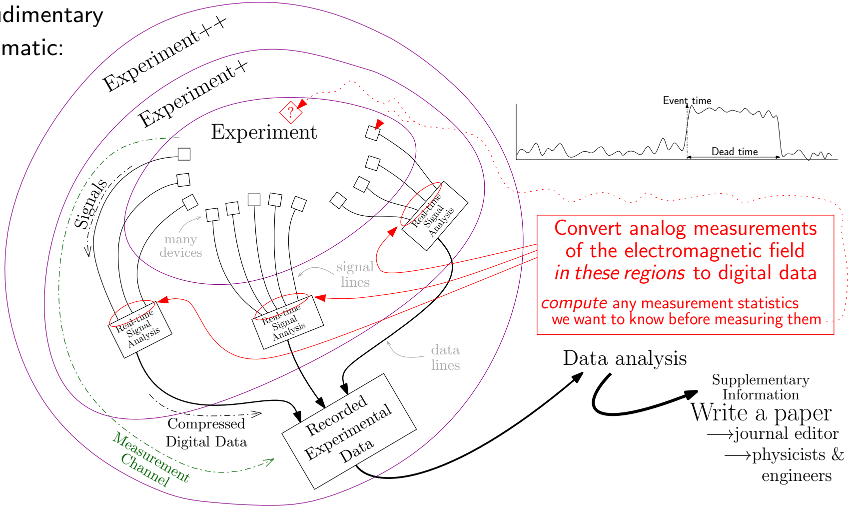

From an *experiment*, we obtain *datasets*, which contain lists of numbers

⟶ average values, relative frequencies, and higher statistics

Something in a theory should generate *expected* averages *etc* for *future* datasets

We can include theoretical objects such as particle properties, but they are extra

what do people do and not do?

- People decide what experiment they want to do
- People find the money to buy the material and parts needed to build the experiment
- People build the apparatus incrementally, checking all the parts work as expected
- People turn on the power

  $\longrightarrow$ The experiment runs, data is collected at GHz rates, and people go for coffee on the Moon

- People decide what computer programs to write to analyze the data
- People write an article about why&how they built the experiment and about its data
- People decide what experiment they want to do next

We could talk about the coffee *etc*,
but my focus is on the GHz-rate data

For a measurement M, with data values $\{m_1, m_2, ...\}$ and relative frequencies $\{p_1, p_2, ...\}$, the weighted average is $\rho(\hat{M}) = \sum_i p_i m_i$

We can write that as $\rho(\hat{M}) = \sum_i p_i m_i = \mathrm{Tr}\left[\begin{pmatrix} p_1 & \cdots & \cdots \\ \vdots & p_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}\right] = \mathrm{Tr}[\hat{\rho}\hat{M}]$

$\hat{\rho} \geqslant 0, \mathrm{Tr}[\hat{\rho}] = 1 \ \Rightarrow \ p_i \geqslant 0, \sum_i p_i = 1$

an elementary and concrete matrix model (of a *Generalized* Probability Theory)

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For the average of M′, $\rho(\hat{M}') = \sum_i p'_i m'_i = \mathrm{Tr}\left[\begin{pmatrix} p'_1 & \cdots' & \cdots' \\ \vdots' & p'_2 & \cdots' \\ \vdots' & \vdots' & \ddots' \end{pmatrix} \cdot \begin{pmatrix} m'_1 & 0 & 0 \\ 0 & m'_2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}\right] = \mathrm{Tr}[\hat{\rho}\hat{M}']$

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Multiple probability spaces are contained in a single formal structure

There is *no* physics in this construction — *there is, in particular, no $\hbar$*

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For traditional classical physics, we use different initial conditions for different 'contexts'
Operator algebras give us more tools for modeling *incompatible experiments*

▸ models of the Kolmogorov axioms appendix

⟵ #1 ⟶

We can begin "A linear operator $\leftrightsquigarrow$ the data values part of a dataset, $\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$",

instead of "A Hilbert space $\leftrightsquigarrow$ a system", as axioms for QM usually begin

There are *abstract* measurements $\hat{M}_1, \hat{M}_2, \hat{M}_3, \ldots, \hat{M}_1 + \hat{M}_2, \ldots, \hat{M}_1 \hat{M}_2, \ldots$

a dataset data values $\leftrightsquigarrow$ a linear operator eigenvalues $\leftrightsquigarrow$ a random variable outcomes in a sample space, noncommutative $\sim$ quantum or commutative $\sim$ classical, associative, distributive, with unit&scalar$\times$

#1

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A *state* ρ maps measurement operators to *expected dataset averages when/if we collect data*

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probability distribution for $\hat{M}_1$ → $\rho(\delta(\hat{M}_1-u))$
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Because *relative frequencies* are *positive*, *normalized*, and *real-valued*,

$$\boxed{\rho(\hat{A}^\dagger\hat{A}) \geq 0,\ \rho(1)=1, \text{ and } \rho(\hat{A}^\dagger)=\rho(\hat{A})^*}$$

where $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, $\mathrm{j}^\dagger = -\mathrm{j}$

and because we can *add relative frequencies*,

von Neumann linearity: $\boxed{\rho(\lambda\hat{A}+\mu\hat{B}) = \lambda\rho(\hat{A})+\mu\rho(\hat{B})}$

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We can also use measurement operators to *modulate* the state ρ to give different expected measurement results,

$$\boxed{\rho_A(\hat{M}) = \frac{\rho(\hat{A}^\dagger\hat{M}\hat{A})}{\rho(\hat{A}^\dagger\hat{A})}}$$

which is the basis of the Gelfand-Naimark-Segal-construction of a Hilbert space

The GNS-construction lets us think of $\rho_v(\hat{M})$ as $\langle v|\hat{M}|v\rangle$ and of $\rho_{Av}(\hat{M})$ as $\dfrac{\langle v|\hat{A}^\dagger\hat{M}\hat{A}|v\rangle}{\langle v|\hat{A}^\dagger\hat{A}|v\rangle}$

"An algebraic approach to Koopman classical mechanics", PM, AnnPhys 2020
Carl Eckart, PhysRev 1926, Bernard Koopman, PNAS 1931

Eckartesian Mechanics? :-P

Take classical mechanics to be an algebra of functions of *position* and *momentum* that has *three* binary operations: addition, multiplication, *and* the Poisson bracket

$$u+v$$
$$u\cdot v$$
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We can construct a simpler algebra of transformations
by introducing "Multiply by w", $\hat{Y}_w(u)=w\cdot u$,
and "Poisson by w", $\hat{Z}_w(u)=\{w,u\}$

$$[\hat{Y}_v,\hat{Y}_w]=0$$
$$[\hat{Z}_v,\hat{Y}_w]=\hat{Y}_{\{v,w\}}\neq 0$$
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A familiar example: "Poisson by the *Hamiltonian function*" gives
a generator of time evolution, $\hat{Z}_H(u)=\{H,u\}$, the *Liouvillian* operator
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I suggest:
We can use the $\hat{Y}$'s *and* $\hat{Z}$'s of a more powerful CM_+ *without restriction*: $\hat{Z}_w^n$, ... (see #11)
That gives us an algebraic measurement theory shared with QM,
including noncommutativity, measurement incompatibility, ...

We can have isomorphisms instead of quantization and the Correspondence Principle

a state for the classical simple harmonic oscillator

The Poisson bracket: $\{v,u\} = \frac{\partial v}{\partial p}\frac{\partial u}{\partial q} - \frac{\partial v}{\partial q}\frac{\partial u}{\partial p}$

$$\hat{Y}_q[u] = q \cdot u, \quad \hat{Z}_p[u] = \{p,u\} = \ \frac{\partial}{\partial q}u, \quad [\hat{Y}_q, \hat{Z}_p] = -1$$

Two copies of the abstract Weyl algebra

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The *Gibbs thermal state* for an SHO at temperature kT is a Normal probability distribution

$$\rho_{\mathrm{Gibbs}}\left(\delta(\hat{Y}_q - \alpha)\delta(\hat{Y}_p - \beta)\right) = \mathrm{e}^{-(\alpha^2+\beta^2)/2\mathsf{kT}}/2\pi\mathsf{kT}$$

The Fourier transform of that probability distribution, using $\hat{Y}_q$ and $\hat{Y}_p$ and the engineer's imaginary j, is $\rho_{\mathrm{Gibbs}}\left(\mathrm{e}^{\mathrm{j}\lambda\hat{Y}_q + \mathrm{j}\mu\hat{Y}_p}\right) = \mathrm{e}^{-\mathsf{kT}(\lambda^2+\mu^2)/2}$

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We can construct modulated, non-equilibrium states, $\dfrac{\langle_{\mathsf{kT}}|\hat{A}^\dagger\,\hat{M}\,\hat{A}|_{\mathsf{kT}}\rangle}{\langle_{\mathsf{kT}}|\hat{A}^\dagger\hat{A}|_{\mathsf{kT}}\rangle} \longrightarrow$ a thermal Hilbert space

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We can construct modulated, non-equilibrium states, $\dfrac{\langle_{\mathrm{kT}}|\hat{A}^\dagger\,\hat{M}\,\hat{A}|_{\mathrm{kT}}\rangle}{\langle_{\mathrm{kT}}|\hat{A}^\dagger\hat{A}|_{\mathrm{kT}}\rangle} \longrightarrow$ a thermal Hilbert space

Instead of trying to map $(q,p) \not\mapsto (\hat{q},\hat{p})$, as quantization tries to *(but fails)*,
we *can* map CM$_+$ to QM, $(q,\mathrm{j}\frac{\partial}{\partial q}) \mapsto (\hat{q}_1,\hat{p}_1)$, $(p,\mathrm{j}\frac{\partial}{\partial p}) \mapsto (\hat{q}_2,\hat{p}_2)$,
however kT is not $\hbar$: thermal noise is not quantum noise

What *is* the difference between quantum and thermal noise?

- $\hbar$ has units of action, whereas kT has units of energy
- In QFT, the quantum vacuum is Lorentz invariant,

  thermal noise is *not* invariant under Lorentz boosts

  This difference of symmetry properties *can* be used in CM_+

  with $\hbar$ as an amplitude of a Lorentz invariant noise

  and with kT as an amplitude of a less invariant thermal noise

quantum noise appendix

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  thermal noise is *not* invariant under Lorentz boosts
  This difference of symmetry properties *can* be used in CM_+
  with $\hbar$ as an amplitude of a Lorentz invariant noise
  and with kT as an amplitude of a less invariant thermal noise

This gives a new reason to think that we must work with field theories, because we can only *define* the Lorentz group in 1+*n*-dimensions

For CM_+, $\hbar \to 0$ is *not* a classical approximation, it's a mean-field approximation

CM_+ can and should include

(1) noncommutativity

(2) quantum noise

$\longrightarrow$ the measurement theory is the same as for QM

CM_+ can and should include
(1) noncommutativity
(2) quantum noise
$\longrightarrow$ the measurement theory is the same as for QM

but there is a more technical difference:
For QM, the Hamiltonian operator is bounded below $\longrightarrow$ analytic properties

For CM_+, in the Gibbs state of the Simple Harmonic Oscillator,
$\mathrm{j}\hat{Z}_H$ is the Hermitian generator of evolution,

$$\mathrm{j}\hat{Z}_H = \mathrm{j}\left(p\cdot\frac{\partial}{\partial q} - q\cdot\frac{\partial}{\partial p}\right) \ngeq 0$$

$\longrightarrow$ ~~analytic properties~~

(3) analyticity is mathematically useful but should *not* be included in CM_+
We can say that QM is an analytic form of CM_+
This is like using *the analytic signal* or *the real signal* in signal analysis

For a classical SHO, we are allowed to use $\hat{Z}_p = \dfrac{\partial}{\partial q}$ to generate translations,

$$\mathrm{e}^{-\kappa \hat{Z}_p} \cdot \hat{Y}_q \cdot \mathrm{e}^{\kappa \hat{Z}_p} = \hat{Y}_q - \kappa = \hat{Y}'_q,$$

but we are *not* allowed to use $\hat{Z}_p^3$ to generate a unitary transformation,

$$\mathrm{e}^{-\kappa \hat{Z}_p^3} \cdot \hat{Y}_q \cdot \mathrm{e}^{\kappa \hat{Z}_p^3} = \hat{Y}_q - 3\kappa \hat{Z}_p^2 = \hat{Y}_q^{\text{very different, not different}}$$

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Quantum mechanics happily uses $\hat{Z}_p^3$ to transform to a different experiment

$\hat{Z}_p^3$ *et cetera can* be useful in the same way for classical mechanics

(as for the elementary matrix model on #5)

If we unlock that superpower for classical mechanics,

we can *unshackle* classical mechanics

the idea of what an initial condition *is* becomes more complicated

An initial condition for CM_+ contains *many* of CM's initial conditions

CM_+ has a measurement problem

"The collapse of a quantum state as a joint probability construction", PM, JPhysA 2022

For a measurement A, with data values $\mathcal{A}=\{\alpha_m\}$, $\hat{A}=\sum_m \alpha_m \hat{P}_m = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$, and a measurement B, with data values $\mathcal{B}=\{\beta_n\}$, $\hat{B}=\sum_n \beta_n \hat{Q}_n$,

For a measurement of A, with state vector $|\psi\rangle$, we obtain the result α_m with probability $p(\alpha_m)=\langle\psi|\hat{P}_m|\psi\rangle$ and similarly for B

for simple cases, $\hat{P}_m$ is a shorthand for $|\alpha_m\rangle\langle\alpha_m| \longrightarrow \langle\psi|\alpha_m\rangle\langle\alpha_m|\psi\rangle = |\langle\psi|\alpha_m\rangle|^2$, which is the Born rule

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For two measurements, of A first, to be followed by B,
we say that the result α_m "collapses" the state vector,

$$|\psi\rangle \longrightarrow |\psi_m\rangle = \frac{\hat{P}_m|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_m|\psi\rangle}},$$ which is the Lüders rule,

CM_+ has a measurement problem

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then we measure B in that state, so we obtain the result α_m followed by β_n
with *conditional* probability

$$p(\beta_n|\alpha_m) = \langle\psi_m|\hat{Q}_n|\psi_m\rangle = \frac{\langle\psi|\hat{P}_m\hat{Q}_n\hat{P}_m|\psi\rangle}{\langle\psi|\hat{P}_m|\psi\rangle},$$

so the *joint* probability is

$$p(\alpha_m \text{ and } \beta_n) = \langle\psi|\hat{P}_m\hat{Q}_n\hat{P}_m|\psi\rangle.$$

We have $p(\alpha_m \text{ and } \beta_n) = \langle\psi|\hat{P}_m\hat{Q}_n\hat{P}_m|\psi\rangle$,

so the positive operators $\hat{J}_{mn} = \hat{P}_m\hat{Q}_n\hat{P}_m$ generate

the joint probabilities $\langle\psi|\hat{J}_{mn}|\psi\rangle$ for the results (α_m, β_n).

Instead of collapse affecting a state,

we can take collapse to affect the next measurement, $\hat{P}_m\hat{Q}_n \longrightarrow \hat{P}_m \cdot \hat{P}_m\hat{Q}_n\hat{P}_m$

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A⧑B, with data values $\{(\alpha_m, \beta_n)\}$, even if $[\hat{A}, \hat{B}] \neq 0$

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A⧓B, with data values $\{(\alpha_m, \beta_n)\}$, even if $[\hat{A}, \hat{B}] \neq 0$

The existence of a joint probability is traditionally "classical": we can indeed
use commuting operators $\hat{A}'$ and $\hat{B}'$ and a *different* vector state $|\psi'\rangle \in \mathcal{H}'$
that give the same joint probability, $\langle\psi'|\hat{P}'_m\hat{Q}'_n|\psi'\rangle = \langle\psi|\hat{P}_m\hat{Q}_n\hat{P}_m|\psi\rangle$
new,'classical' $\longleftarrow$ original, quantum

Mathematically, this is to use the Naimark Dilation Theorem to construct
a joint measurement $\widehat{AB}$ that is the same as A⧓B (for a larger Hilbert space),
which can be an *alternative* to "Decoherence"

‡ 'Naimark' has an alternate spelling, 'Neumark'

Decoherence takes a *large* Hilbert space model
for *whatever is measured* together with *whatever measures it* and
shows that environmental noise will *very nearly* "collapse" the state
to an after-measurement state

For All Practical Purposes Decoherence works well enough

Naimark's Dilation Theorem[‡] as an *alternative* to Decoherence

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Naimark is somewhat the opposite

Naimark takes a *small* Hilbert space model for *whatever is measured* and
constructs a joint probability for consecutive measurements
$\longrightarrow$ a model for a measurement device in a larger Hilbert space
in the context of earlier and subsequent measurements

See also *The Principle of Deferred Measurement*

Naimark constructs a rough model and refines it as needed

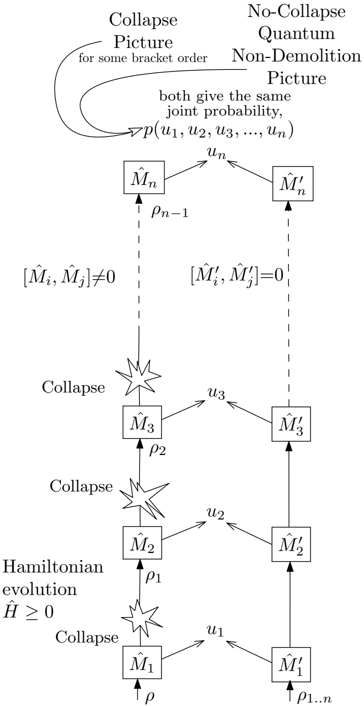
as classical physics and engineers always did

If the datasets we have don't tell us enough, get more data to analyze

We can use Decoherence or Naimark, whichever seems easier, more appropriate, ...

For signal analysis,
there are *many* measurements
at timelike separation

We can use $\hat{M}_1, ..., \hat{M}_{100...000}$,
with many collapses,
or we can use $\hat{M}'_1, ..., \hat{M}'_{100...000}$,
which all commute, with no collapses



#1

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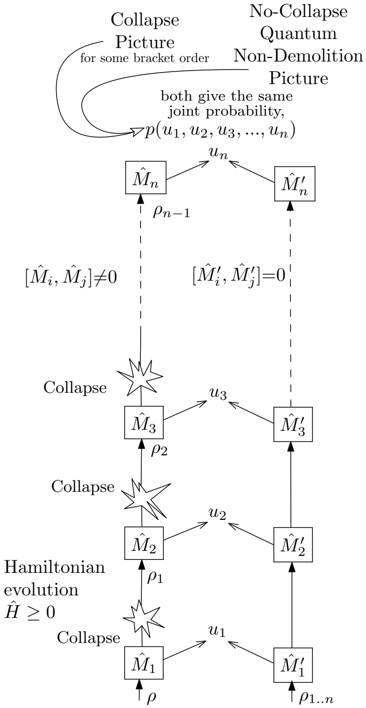
"Collapse" is not
(only or necessarily)
a dynamical process

We can (also) take it to be a
JOINT PROBABILITY ALGORITHM

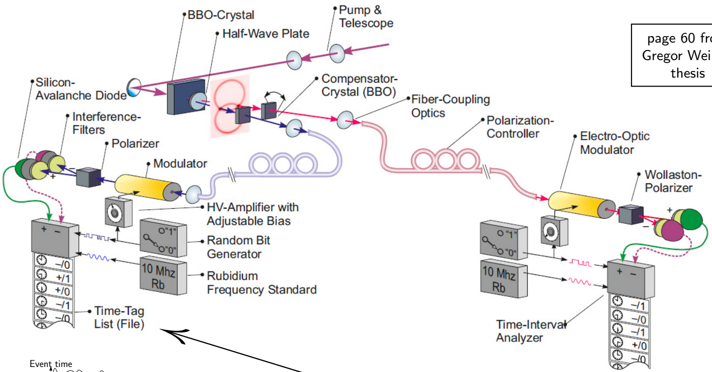


Think in terms of a Noisy classical field and signal analysis, not particles

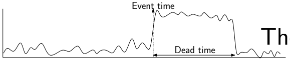
As for the heat equation, boundary conditions matter differently than they do for particles

A central apparatus modulates the ground state; Alice and Bob both have

two Avalanche PhotoDiodes, an Electro-Optic Modulator, a Random Bit Generator, and a clock



page 60 from Gregor Weihs's thesis



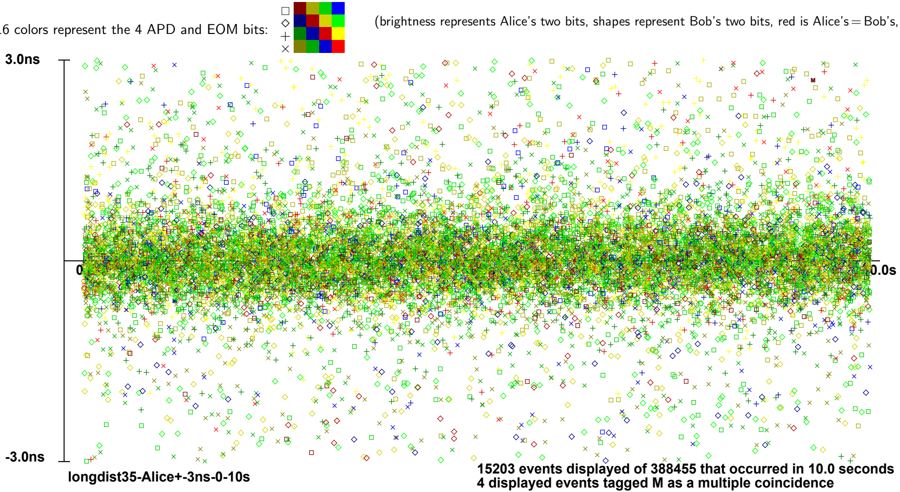
The time when an APD's signal rises to a higher level is recorded, and which APD it was, and what the EOM setting was: when and 2 bits

Gregor gets measurement results I

Alice sees almost 400,000 APD events in 10 seconds
Alice & Bob see over 15,000 coincident event pairs

16 colors represent the 4 APD and EOM bits:

(brightness represents Alice's two bits, shapes represent Bob's two bits, red is Alice's = Bob's, ...)



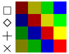
For over 15,000 of Alice's almost 400,000 events, Bob also records an event within 3 nanoseconds

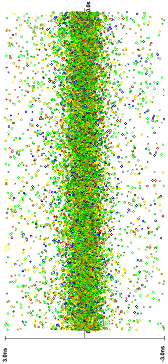
When Alice and Bob both record an event within 3 nanoseconds, the majority are green or yellow

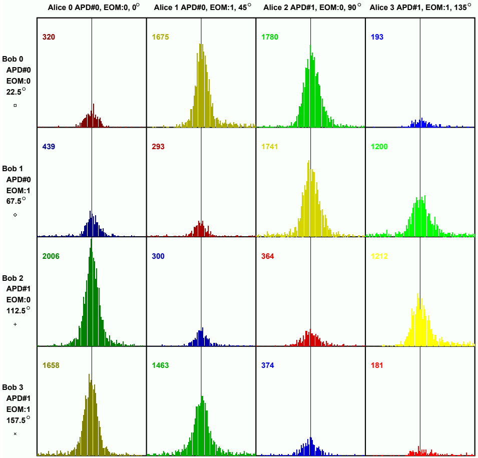
Gregor gets measurement results II

Alice sees almost 400,000 APD events in 10 seconds
Alice & Bob see over 15,000 coincident event pairs

16 colors represent the 4 APD and EOM bits:







E00 = -0.694
[320+364,-2006-1780]
E01 = -0.614
[439+374,-1658-1741]
E10 = 0.708
[1675+1212,-300-193]
E11 = -0.698
[293+181,-1463-1200]

$$|E00+E01-E10+E11| = 2.714$$

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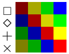

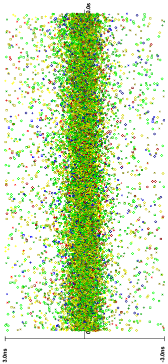

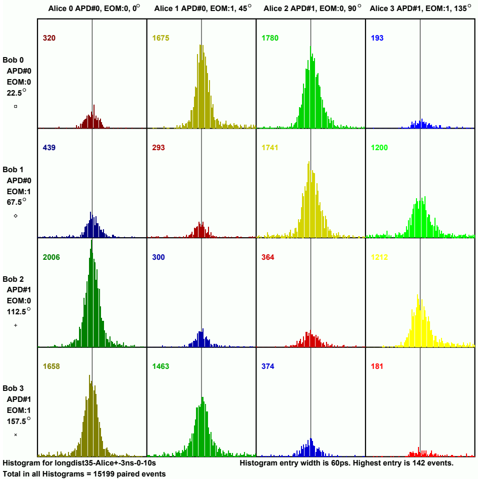

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$$|E00+E01-E10+E11| = 2.714$$

QM uses noncommuting operators to model Bell-violating statistics

CM has only commuting operators, so it's *computationally incomplete*

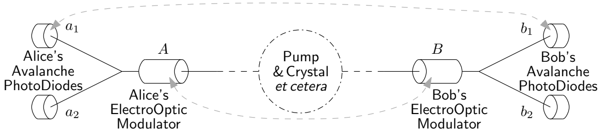
But we have added noncommutativity into CM_+ (it's just Hilbert spaces, etc)

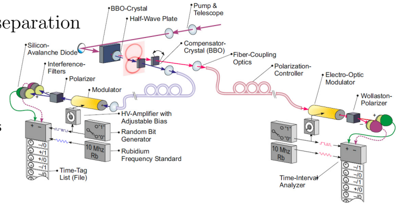
Fine 1982 Landau 1987
PM, AnnPhys 2020, §7.2

cf "Bell inequalities for random fields", PM, JPhysA 2006

We engineer the central device so that, when the power is on, distant *events* violate a Bell inequality

Events *are* coincident ⇒ there *are* significant correlations at space-like separation





engineering *noise* in the whole apparatus

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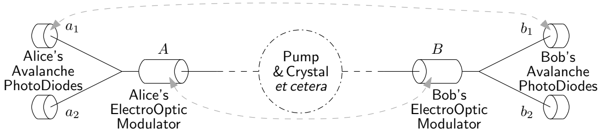

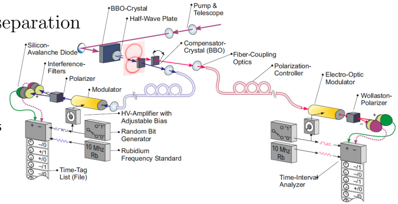

We make sure the EOM *settings* from the two random bit generators are very little correlated, *but* we should expect correlations between the inner workings of the two EOMs

1. there *are* correlations between the inner workings of the four APDs
2. the EOMs act on the noisy EM field ⇒ the noisy EM field also acts on the EOMs

⟶ with those correlations, Bell inequalities cannot be derived

#1

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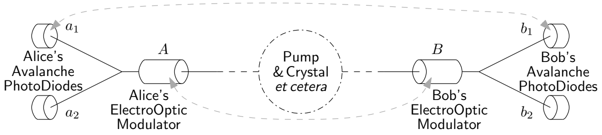

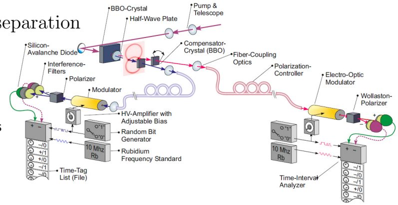

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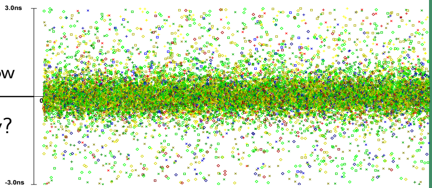
power off very few events	lots of events	lots of events and coincidences	lots of events, coincidences, and green&yellow



Do events, coincidences, and green&yellow propagate differently?
What changes as we change the distance?
This *probes* nonlocal propagation experimentally

If the datasets we have don't tell us enough, get more data to analyze

#1

A new interpretation changes nothing — except our intuition

It doesn't change what an experiment does

It changes how we describe and imagine what an experiment does

It changes what experiment we think it would be interesting to do next

Existing interpretations and textbooks for QM still work just fine —

it would be a big mistake to discard quantization & all that

There are no textbooks for CM_+

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classical mechanics		generalized mechanics	
classical probability		generalized probability	(as on #5)
classical system	⟶	generalized system	
classical subsystem		generalized subsystem	
classical particle		generalized particle	

A small suggestion: say 'generalized' instead of 'quantum'

Part II

In QFT textbooks, for a scalar field, $\hat{\phi}(x)=\int\left[a(k)\mathrm{e}^{-\mathrm{j}k\cdot x}+a^\dagger(k)\mathrm{e}^{\mathrm{j}k\cdot x}\right]\frac{\mathrm{d}^4k}{(2\pi)^4}$

$\hat{\phi}(x)$ is *not* a measurement operator, it's an *operator-valued distribution*,

but we construct $\hat{M}_f=\int\hat{\phi}(x)f(x)\mathrm{d}^4x$, $f(x)$ is a $\left\{\begin{matrix}\text{smearing}\\ \text{test}\\ \text{window}\end{matrix}\right\}$ function

$f(x)$ tells us where, how big, and what shape the '*f-measurement*' is

We can write $\hat{M}_f$ as two components, $\hat{M}_f=a_{f^*}+a_f^\dagger$

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For quantum mechanics, we had measurement operators $\hat{M}_1, \ldots, \hat{M}_n$

To make contact with experiment, we need $\mathit{Description}_1, \ldots, \mathit{Description}_n$,

Anticipating QFT, we can write $\hat{M}_{\mathit{Description}_1}, \ldots, \hat{M}_{\mathit{Description}_n}$

For QFT, $\hat{M}_f = \int \hat{\phi}(x) f(x) \mathrm{d}^4 x$ gives us measurement operators $\hat{M}_{f_1}, \ldots, \hat{M}_{f_n}$

The functions $f_1, \ldots, f_n$ are formal, idealized *Descriptions* of how measurements are not point-like

QFT is the same as QM — except that for each measurement it adds an idealized description of what the measurement does

For the free quantized Klein-Gordon field, with $\hat{M}_f = a_{f^*} + a_f^\dagger$,

$[a_f, a_g^\dagger] = (f, g)$ is a *pre*-inner product:

(f, g) can be zero when f and g are both non-zero

We can call (f, g) the *overlap* of f and g:

if $(f, f) = 1$ and $(g, g) = 1$, then $0 \leq |(f, g)| \leq 1$

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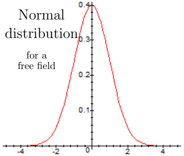
We can calculate probabilities if all we know is

$$[a_f, a_g^\dagger] = (f, g),\ [a_f, a_g] = 0,$$
$$a_f|v\rangle = 0, \quad \hat{M}_f = a_{f^*} + a_f^\dagger$$

If $f^* = f$, $\langle v|\delta(\hat{M}_f - u)|v\rangle = \dfrac{\mathrm{e}^{-u^2/2(f,f)}}{\sqrt{2\pi(f,f)}}$ is a Normal probability distribution with variance (f, f)

because, using a Baker-Campbell-Hausdorff identity, $\langle v|\,\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}|v\rangle = \mathrm{e}^{-\lambda^2(f^*,f)/2}$



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(f,g) can be zero when f and g are both non-zero

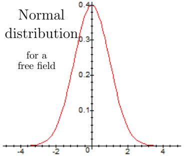
We can call (f,g) the *overlap* of f and g:

if $(f,f)=1$ and $(g,g)=1$, then $0 \le |(f,g)| \le 1$

We can calculate probabilities if all we know is

$$[a_f, a_g^\dagger] = (f,g),\ [a_f, a_g] = 0,$$
$$a_f|v\rangle = 0, \quad \hat{M}_f = a_{f*} + a_f^\dagger$$

If $f^* = f$, $\langle v|\delta(\hat{M}_f - u)|v\rangle = \dfrac{\mathrm{e}^{-u^2/2(f,f)}}{\sqrt{2\pi(f,f)}}$ is a Normal probability distribution with variance (f,f)



because, using a Baker-Campbell-Hausdorff identity, $\langle v|\,\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}\,|v\rangle = \mathrm{e}^{-\lambda^2(f^*,f)/2}$

Modulated states, $\dfrac{\langle v|\hat{M}_f^\dagger\,\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}\,\hat{M}_f|v\rangle}{\langle v|\hat{M}_f^\dagger\hat{M}_f|v\rangle}$, ..., give modulated probability distributions

and a usefully smaller Hilbert space than the full Fock space

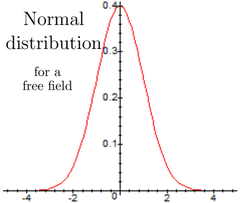
Quantum Optics is a field theory that uses only a few different window functions

⟵ #1 ⟶

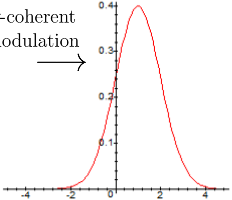
Think of the vacuum state of a quantum field as
a broadband, noisy "carrier signal" for *probabilistic* modulations
of measurement results

Normal distribution

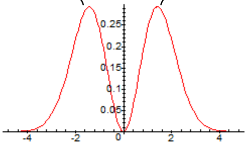
for a free field



g-coherent modulation $\longrightarrow$



g-raised modulation $\longleftarrow$ $\longrightarrow$



$$\rho_v(\delta(\hat{M}_f - u)) = \frac{\mathrm{e}^{-u^2/2(f,f)}}{\sqrt{2\pi(f,f)}} \qquad \frac{\mathrm{e}^{-(u-((g,f)-(f,g))/\mathrm{j})^2/2(f,f)}}{\sqrt{2\pi(f,f)}} \qquad \text{(omitted)}$$

probability distributions

$$\rho_v(\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}) = \mathrm{e}^{-\lambda^2(f,f)/2} \qquad \rho_v(\mathrm{e}^{-\mathrm{j}\hat{M}_g}\,\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}\,\mathrm{e}^{\mathrm{j}\hat{M}_g}) \qquad \frac{\rho_v\left(\hat{M}_g^\dagger\,\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}\,\hat{M}_g\right)}{\rho_v\left(\hat{M}_g^\dagger\hat{M}_g\right)}$$

characteristic functions

$\hat{M}_f^\dagger = \hat{M}_{f^*}$ and $\rho_v(\hat{M}_f^\dagger\hat{M}_g) = (f, g)$ determine the geometric structure of a free quantum field
tell us how a modulation changes measurement results

quantum fields — modulation & measurement

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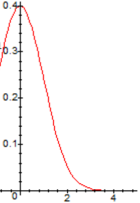

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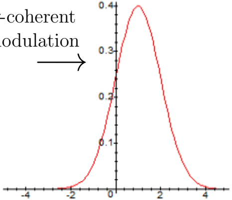

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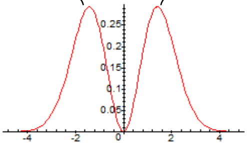

$$\rho_v(\delta(\hat{M}_f - u)) = \frac{e^{-u^2/2(f,f)}}{\sqrt{2\pi(f,f)}} \qquad \frac{e^{-(u-((g,f)-(f,g))/j)^2/2(f,f)}}{\sqrt{2\pi(f,f)}} \qquad \text{(omitted)}$$

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$$\rho_v(e^{j\lambda\hat{M}_f}) = e^{-\lambda^2(f,f)/2} \qquad \rho_v(e^{-j\hat{M}_g}\, e^{j\lambda\hat{M}_f}\, e^{j\hat{M}_g}) \qquad \frac{\rho_v\left(\hat{M}_g^\dagger\, e^{j\lambda\hat{M}_f}\, \hat{M}_g\right)}{\rho_v\left(\hat{M}_g^\dagger \hat{M}_g\right)}$$

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tell us how a modulation changes measurement results

We can also modulate joint measurements:
as Gregor Weihs's experiment did

$$\frac{\rho_v(\hat{A}^\dagger e^{j\lambda_1\hat{M}_{f_1}}\, e^{j\lambda_2\hat{M}_{f_2}} \cdots \hat{A})}{\rho_v(\hat{A}^\dagger\hat{A})}$$

Call this a "Wigner-characteristic function"

The Wigner-characteristic function gives an effective presentation of the algebraic structure

$$\rho_v(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}_{f_2}}\cdots)=\rho_v(\mathrm{e}^{\sum_i\mathrm{j}\lambda_i\hat{M}_{f_i}})\exp\Big[-\sum_{i<j}\lambda_i\lambda_j[\hat{M}_{f_i},\hat{M}_{f_j}]/2\Big]$$

for the Gaussian case

$$=\exp\Big[-\sum_{i,j}\lambda_i\lambda_j(f_i^*,f_j)/2-\sum_{i<j}\lambda_i\lambda_j[(f_i^*,f_j)-(f_j^*,f_i)]/2\Big]$$

Gaussian noise term NONCOMMUTATIVITY TERM

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We can fix the geometric & dynamical structure in multiple ways:

Klein-Gordon: $(f,g)=\hbar\int\tilde{f}^*(k)\tilde{g}(k)2\pi\delta(k\cdot k-m^2)\theta(k_0)\frac{\mathrm{d}^4k}{(2\pi)^4}$

Quantum Optics: $(f,g)=-\hbar\int\underbrace{\tilde{f}^*_{\alpha\mu}(k)k^\mu\, k^\nu\tilde{g}^\alpha{}_\nu(k)}_{\text{two space-like 4-vectors}}2\pi\delta(k\cdot k)\theta(k_0)\frac{\mathrm{d}^4k}{(2\pi)^4}$

both of which are positive-frequency and Lorentz and translation invariant

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so remove the “$\theta(k_0)$”, which gives us

an *everywhere commutative*, Noisy classical field theory,

with a Lorentz and translation invariant $\hbar$-scale noise

For the Gaussian state of Quantum Optics, using the (f,g) on #24

$$\boxed{\begin{array}{c}[a_f, a_g^\dagger] = (f,g),\ [a_f, a_g] = 0, \\ a_f|v\rangle = 0, \quad \hat{M}_f = a_{f^*} + a_f^\dagger\end{array}} \longrightarrow [\hat{M}_f, \hat{M}_g] = (f^*, g) - (g^*, f) \text{ is sometimes non-zero}$$

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For Quantum Optics, we can find an involution $f\mapsto f^\bullet$, $f^{\bullet\bullet}=f$,

for which $\underline{(f^{*\bullet},g^\bullet)-(g^{*\bullet},f^\bullet)=0}$, for all test functions f and g

For $\hat{\mathtt{M}}_f^{\text{QND}}=a_{f^{*\bullet}}+a_{f^\bullet}^\dagger\neq\hat{M}_{f^\bullet}$, $\quad [\hat{\mathtt{M}}_f^{\text{QND}},\hat{\mathtt{M}}_g^{\text{QND}}]=0$

The $\hat{\mathtt{M}}_f^{\text{QND}}$ generate an effectively classical QND Optics field:

a commutative algebra of Quantum Non-Demolition measurements

and an isomorphic Hilbert space

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For quantum optics: $\widetilde{f^\bullet}(k)=\frac{1}{2}(1+\mathrm{j}\star)\tilde{f}(k)+\frac{1}{2}(1-\mathrm{j}\star)\tilde{f}(-k)$

$f\mapsto f^\bullet$ is Lorentz invariant but *not* translation invariant or local,

but both Quantum Optics and QND Optics

are Lorentz invariant and translation invariant

quantum and QND Optics II

The algebra generated by the $\hat{\mathtt{M}}_f^{\text{QND}}$ is *not* isomorphic to that generated by the $\hat{M}_f$,
but with an *unrestricted* Poisson bracket —*or* with $\updownarrow$-operators *or* with $|v\rangle\langle v|$—
anything we can model with Quantum Optics,
we can also model with QND Optics

- QFT: $\hat{M}_f$ *sometimes* commutes with $\hat{M}_g$
- QNDFT: $\hat{\mathtt{M}}_f^{\text{QND}}$ *always* commutes with $\hat{\mathtt{M}}_g^{\text{QND}}$
  but we use generalized probability as well, when it's needed — it's inside QFT

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Signal Analysis +1+3-dimensions+algebraic generalized probability+quantum noise
Stochastic Processes+1+3-dimensions+algebraic generalized probability+quantum noise
like de Broglie-Bohm (for QFT), but linear, Lorentz invariant, and more measurement theoretic
NOT like superdeterminism because it's generalized probability at all scales

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I think of $\hat{M}_f$ and $\hat{\mathtt{M}}_f^{\text{QND}}$ as different kinds of idealized measurements
quantum and classical

We never could get data for classically ideal measurements
but we imagined and talked about them as if we could

Actually recorded measurement results (adopt a 'classical' starting point $\sim$ Copenhagen)
$\longrightarrow$ Data Analysis & Predicting Data (Metadata, *Measurement Description*, is *vital*)
Statistics & Expected Statistics (+classical measurement incompatibility)

Classical Mechanics+noncommutativity+quantum noise $\longrightarrow$
Quantum and Classical$_{+/\text{QND}}$ are types of description, not types of system

There is no such thing as "nonclassicality" *if* we allow CM$_+$

"Quantum" just means quantum noise can't be ignored

We can think of renormalization as a surreptitious
introduction of nonlinearity into the Wightman axioms
We can construct a more empiricist QG by focusing on metadata

Quantum and Classical have been
converging, in numerous ways, for decades

Generalized Probability Theories, phase space methods, contextuality, non-demolition measurement,
Koopman CM, time-frequency analysis, stochastic methods, semi-classical methods, superdeterminism,
causal modeling, Cohen 1988 on characteristic functions, Abramsky 2020 on Boole's "Conditions of Possible Experience", ...

#1

A Dataset & Signal Analysis Interpretation of Quantum Mechanics, Bhadrak, March 1st, 2025

Part I suggests that it's helpful to think of Quantum Mechanics as about Datasets and Signal Analysis *as well as* about quantum particles and their properties. Signal analysis is more powerful than Classical Mechanics because it allows noncommutativity, like Quantum Mechanics, which encourages us to look for a way to create a more powerful 'CM_+' that includes noncommutativity (spoiler#1: we can do that easily by using the Poisson bracket.) If we start to think in that generalized classical way, then we need a mathematically well-motivated distinction between thermal fluctuations and quantum fluctuations (spoiler#2: we can find a good answer, Lorentz invariance, in quantum thermodynamics.) The introduction of CM_+ also allows us to rethink the measurement problem in a generalized classical setting as a way to construct joint measurement probabilities even when measurements are incompatible.
As a transition to the second part, we can discuss Bell inequalities entirely in a noisy classical field framework, which suggests that Classical Mechanics as usually presented is *computationally incomplete* whereas the introduction of generalized probability makes CM_+ more complete.

Part II takes Quantum Field Theory to be a noisy signal analysis formalism. For the quantized electromagnetic field case we can construct a Quantum Non-Demolition Field Theory (a QNDFT) as a noisy classical field embedded within QFT, which allows us, if we're careful, to think in a more classical way about physics.

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Part III (not today) The signal analysis perspective allows us to rethink the renormalization 'problem' as a surreptitious introduction of nonlinearity into axiomatic QFT, which allows a reinvention of interacting QFT as a model of the Wightman axioms with a single nonlinear modification. Signal analysis also allows us to construct a more empiricist approach to Quantum Gravity by focusing more on the 'window functions' that describe a measurement than on the metaphysics that is discovered by measurement.

Eckart 1926, Koopman 1931, Sudarshan 1976

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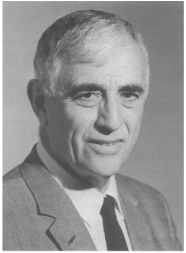
Bernard Osgood Koopman

VOL. 17, 1931 *MATHEMATICS: B. O. KOOPMAN* 315

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

BY B. O. KOOPMAN

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated March 23, 1931

In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential rôle is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics, or, more generally, systems defining a steady n-dimensional flow of a fluid of positive density.

"Questions of ergodicity had been in the foreground for many years and had attracted the attention of powerful mathematicians. Koopman was well versed in this domain and had discussed it with both Birkhoff and von Neumann. In March of 1931, Koopman published a note in the National Academy Proceedings, transforming the problem into one dealing with one parameter unitary groups in Hilbert space."
Von Neumann quickly proved the ergodic theorem in a Hilbert space sense and Birkhoff established point-wise convergence almost everywhere.

"In Memoriam: Bernard Osgood Koopman, 1900-1981", *Operations Research* 1982.

$\longrightarrow$ almost nothing $\longrightarrow$ Sudarshan 1976 / Koopman methods

Earlier, Carl Eckart, "Operator Calculus and the solution of the equations of quantum dynamics", Heilbron&Rovelli 2023 *Phys. Rev.* 1926

Three ways to compare QM and CM: Wigner functions, Hilbert spaces, or algebraic
Koopman Eckart

From a Data Analysis perspective, *for a dataset*, $\underline{not}$ for an ensemble of 'systems',

A Relative Frequency is an integer ratio $\frac{n}{N(C)}, 0 \le n \le N(C)$ in a context C

Adding RFs in the same context with no double counting $\longrightarrow$ a new RF

Adding RFs in the same context with no double counting and no omissions $\longrightarrow 1$

We use sample spaces&σ-algebras to keep track of double counting and omission

Historically, the Kolmogorov axioms followed the mathematics of relative frequencies,

RFs give us a model of the Kolmogorov axioms over $\mathbb{Q}$

Probabilities give us a model of the Kolmogorov axioms over $\mathbb{R}$

We can use algebraic models of the Kolmogorov axioms over $\mathbb{R}$

models of the Kolmogorov axioms

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Generalized Probability gives us ways to interpolate between different contexts

From $\frac{n_1(C_1)}{N(C_1)}, \frac{n_2(C_1)}{N(C_1)}, \ldots; \frac{n_1(C_2)}{N(C_2)}, \ldots; \frac{n_1(C_K)}{N(C_K)}, \ldots$ and by *placing* C_{K+1}

relative to earlier contexts, we can infer expected values for new RFs $\frac{n_i(C_{K+1})}{N(C_{K+1})}$

Generalized Probability Theory gives us more tools

▸ elementary matrix model page

In QFT, the quantum vacuum is Poincaré invariant, thermal noise is not

$$\langle v|\hat{\phi}(x)\hat{\phi}(y)|v\rangle = \hbar\int \qquad 2\pi\delta(k\cdot k - m^2)\theta(k_0)\mathrm{e}^{-\mathrm{j}k\cdot(x-y)}\frac{\mathrm{d}^4k}{(2\pi)^4} \qquad \text{[real K-G QFT]}$$

$$\rho_{\mathrm{kT}}\Big(\hat{\phi}(x)\hat{\phi}(y)\Big) = \hbar\int \coth\left(\tfrac{\hbar k_0}{2\mathrm{kT}}\right) 2\pi\delta(k\cdot k - m^2)\theta(k_0)\mathrm{e}^{-\mathrm{j}k\cdot(x-y)}\frac{\mathrm{d}^4k}{(2\pi)^4}$$

A thermal state for a qSHO is just *extra noise* because $\hat{H}$ is the number operator,

$$[a, a^\dagger] = 1, \qquad \hat{H} = a^\dagger a, \qquad \hat{M} = a + a^\dagger,\ \hat{M}' = \mathrm{j}(a^\dagger - a)/2,\ [\hat{M}, \hat{M}'] = \mathrm{j}$$

$$\rho_v\left(\mathrm{e}^{\mathrm{j}\lambda\hat{M}}\right) = \mathrm{e}^{-\lambda^2/2}$$

$$\rho_{\mathrm{kT}}\left(\mathrm{e}^{\mathrm{j}\lambda\hat{M}}\right) = \frac{\mathrm{Tr}[\mathrm{e}^{\mathrm{j}\lambda\hat{M}}\mathrm{e}^{-a^\dagger a/\mathrm{kT}}]}{\mathrm{Tr}[\mathrm{e}^{-a^\dagger a/\mathrm{kT}}]} = \frac{\mathrm{Tr}[\mathrm{e}^{\mathrm{j}\lambda a^\dagger}\mathrm{e}^{\mathrm{j}\lambda a}\mathrm{e}^{-a^\dagger a/\mathrm{kT}}]}{\mathrm{Tr}[\mathrm{e}^{-a^\dagger a/\mathrm{kT}}]}\mathrm{e}^{-\lambda^2/2} = \cdots$$

see this URL

$$= \exp\left[-\lambda^2/2 - \lambda^2\sum_{n=1}^{\infty}\mathrm{e}^{-n/\mathrm{kT}}\right] = \exp\left[-\lambda^2/2 - \frac{\lambda^2\mathrm{e}^{-1/\mathrm{kT}}}{1-\mathrm{e}^{-1/\mathrm{kT}}}\right]$$

$$= \mathrm{e}^{-\coth(1/2\mathrm{kT})\lambda^2/2}$$

quantum noise page

measurement incompatibility and quantum noise as a cause

If $\hat{M}_f$ and $\hat{M}_g$ are $\begin{smallmatrix}\text{space-like}\\ \text{causally}\end{smallmatrix}$ separated, they are *compatible* and $\frac{\rho(\hat{M}_f\hat{M}_g)}{\sqrt{\rho(\hat{M}_f^2)\rho(\hat{M}_g^2)}}$ is a correlation,

otherwise they are *incompatible* and $\frac{\rho(\hat{M}_f\hat{M}_g)}{\sqrt{\rho(\hat{M}_f^2)\rho(\hat{M}_g^2)}}$ has an imaginary component

The imaginary part is non-zero only at time-like separation

measurement incompatibility and quantum noise as a cause

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The imaginary part is non-zero only at time-like separation

We use *collapse* to construct correlations at time-like separation, but

we can also construct a QNDFT that is empirically equivalent,

for which $\hat{\mathbb{M}}_f^{\text{QND}}$ and $\hat{\mathbb{M}}_g^{\text{QND}}$ are *always* compatible (then we add transformations to obtain noncommutativity)

QFT folds into the formalism *only* the quantum noise aspect of causality, with amplitude $\hbar$, leaving other aspects of causality as separate concerns

For QG, causality may be dynamic, so a QNDFT *convention* may be better

the super-Heisenberg picture

	Apply unitary evolution to the state	Apply collapse to the state	Apply unitary evolution to measurements	Apply collapse to measurements
Schrödinger picture	×	×		
Heisenberg picture		×	×	
super-Heisenberg picture			×	×

This is not a unitary equivalence, but it is an empirical equivalence because we can model the same expected joint statistics

I find this *somewhat* satisfying as a unification *of sorts*, insofar as we can also call it

the *"Bohr picture"*, because it's rather classical and, for Bohr, measurements affect other measurements[†]

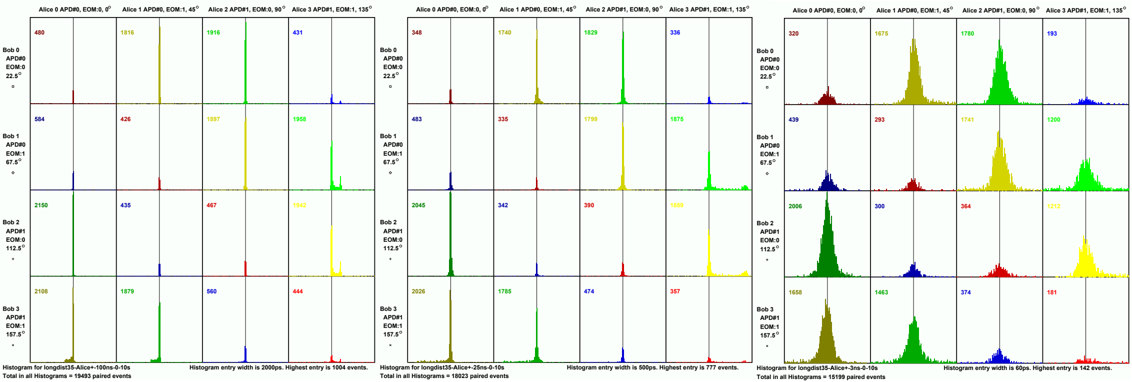
or the *"QND picture"* or the *"Consistent Histories picture"*, because it's commutative

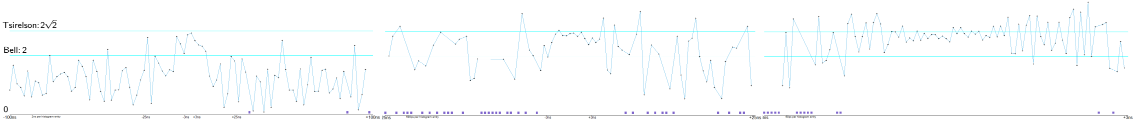
or the *"Everett picture"*, because it's no–collapse (but branching is not *necessary*)

or the *"Einstein picture"*, because it's rather classical (but with a Poincaré invariant noise)

[†] Howard 2004

Gregor's measurement results: more analysis





To the right: ±3 nanosecond histograms ; below it: the Bell-CHSH inequality calculated for every bar of the 16 histograms

At center: ±25 nanosecond histograms — with a subsidiary peak at about ±20 nanoseconds

To the left: ±100 nanosecond histograms

to know why we would have to have the apparatus available, to *debug* it

▸ whole apparatus conditioning page

#1

Test function space

Many engineering and physics applications do not need or want
the mathematical complexity of a representation of the Poincaré group

Using only a small number of test functions is a commonplace in Quantum Optics
An axiomatic approach should not *exclude* a standard engineering practice
Manifest Poincaré invariance can be enough

Use a finite set of test functions as a starting point,
$\mathcal{A}(\{f_1, f_2, \ldots, f_N\})$, generated by $\hat{M}_{f_1}, \hat{M}_{f_2}, \ldots, \hat{M}_{f_N}$, where $f_i \in \mathcal{S}(\mathcal{M})$,
making QFT the same as algebraic QM unless we take an $N \to \infty$ limit

If we need a representation of the Poincaré group,
we can use a direct or inductive limit over Schwartz space

▸ nonlinear axioms page

#1

Bounded operators

For a finite number of test functions, $\mathcal{A}(\{f_1, f_2, \ldots, f_N\})$, the mathematics is just QM, so we can use our Rigged Hilbert space experience for unbounded operators

The vacuum state typically has tails that diminish rapidly enough that the representation of an unbounded algebra is mathematically well-controlled

For the free field vacuum state for $\mathcal{A}(\{f\})$, $\rho_v(\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}) = \mathrm{e}^{-\lambda^2(f^*,f)/2}$,

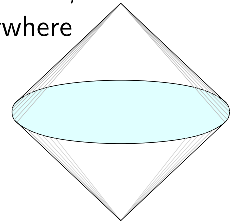
where $(f,g) = \rho_v(\hat{M}_f^\dagger \hat{M}_g)$

We allow unbounded operators $\hat{M}_{f_1}$, $\hat{M}_{f_2}$, ..., $\hat{M}_{f_N}$, as for the Wightman axioms, but we *can* restrict to using only $f(x) \in \mathbb{R}$ and bounded generating operators $\mathrm{e}^{\mathrm{j}\lambda\hat{M}_f}$

nonlinear axioms page

Time-slice axiom

If we know all expected measurement results on a Cauchy surface,
that determines all expected measurement results everywhere
within forward and backward light-cones



For engineering, we never know anything like that much

Given what we know for measurements $\hat{M}_{f_1}, \hat{M}_{f_2}, \ldots, \hat{M}_{f_N}$,
and knowing the relationships between $\hat{M}_{f_1}, \hat{M}_{f_2}, \ldots, \hat{M}_{f_N}$, *and* $\hat{M}_{f_{N+1}}$
we want a manifestly invariant vacuum state of a theory to give
conditional probabilities for another measurement $\hat{M}_{f_{N+1}}$,

This construction is neutral about reductionism
because $f_1, f_2, \ldots, f_N$ and f_{N+1} can all be of arbitrary scale cf Adlam 2024

nonlinear axioms page

#1

fermionic and ElectroMagnetic measurements

If a theory tells us *everything* about a deformed ElectroMagnetism, at *all* length scales, that tells us *everything* about the currents

For an interacting EM field, $\hat{M}'_f \neq \hat{M}_f$, $\rho'_v(\mathrm{e}^{\mathrm{j}\lambda_1 \hat{M}'_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2 \hat{M}'_{f_2}}\cdots) \neq \rho_v(\mathrm{e}^{\mathrm{j}\lambda_1 \hat{M}_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2 \hat{M}_{f_2}}\cdots)$, (interacting / free)
we might say that $\hat{M}'_f$ introduces fermionic and other "clouds", but,
$\rho'_v(\mathrm{e}^{\mathrm{j}\lambda_1 \hat{M}'_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2 \hat{M}'_{f_2}}\cdots)$ still must be manifestly Poincaré invariant
in terms of *just* the λ's and f's

Electron, Proton, and other nontrivial modulations of the EM field are *useful*, but they may be not *necessary*

▸ fermionic measurements appendix

fermionic measurements

We can construct measurement operators using a Dirac fermion test function,

$$\hat{M}_U = \hat{\psi}_U^\dagger \hat{\psi}_U, \text{ where } \{\hat{\psi}_U^\dagger, \hat{\psi}_V\} = (U,V),\ \{\hat{\psi}_U, \hat{\psi}_V\} = 0$$

$$\frac{\hat{M}_U}{(U,U)} \text{ is a projection operator, } \hat{M}_U^n = (U,U)^{n-1}\hat{M}_U$$

$$\rho_\mathrm{v}(\hat{M}_U) = (U,U)_+ \leqslant (U,U)$$

$$\rho_\mathrm{v}(\mathrm{e}^{\mathrm{j}\lambda\hat{M}_U}) = \left[1 - \frac{(U,U)_+}{(U,U)}\right] + \frac{(U,U)_+}{(U,U)}\mathrm{e}^{\mathrm{j}\lambda(U,U)}$$

$$\rho_\mathrm{v}(\delta(\hat{M}_U - \alpha)) = \left[1 - \frac{(U,U)_+}{(U,U)}\right]\delta(\alpha) + \frac{(U,U)_+}{(U,U)}\delta(\alpha - (U,U))$$

$\hat{M}_U$ is a nonlinear majority rule-type construction

The general case, $\rho_\mathrm{v}(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}_{U_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}_{U_2}}\cdots)$, is not straightforward

We *could* introduce $\hat{M}_{(f,U)} = \hat{M}_f + \hat{M}_U$, with EM and Dirac components

‣ fermionic and ElectroMagnetic measurements appendix

#1

As for any experiment, a Generalized Computer does what it does,
whether we describe it using QM or CM_+
Generalized (or Quantum) Computing is a kind of hardware acceleration
that is worthwhile for a specific class of algorithms

Existing Quantum Computing textbooks are unaffected

For *Generalized Computing*, we engineer less constrained hardware, with generalized systems as an *abstract layer* for noisy field theoretic models

Discretization events are chaotically timed,
instead of coordinated by a clock signal,
but we engineer events to have correlated timings

Part III (not today)

a prelude to interacting fields

For a Gaussian field, the Wigner-characteristic function fixes the algebraic structure:

$$\rho_v(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}_{f_2}}\cdots)=\exp\Big[\underbrace{-\sum_{i,j}\lambda_i\lambda_j(f_i^*,f_j)/2}_{\text{Gaussian noise term}}\underbrace{-\sum_{i<j}\lambda_i\lambda_j[(f_i^*,f_j)-(f_j^*,f_i)]/2}_{\text{NONCOMMUTATIVITY TERM}}\Big]$$

For an interacting field, we want to deform that functional of the dataset descriptions f_i, carefully keeping ρ_v still as a state for the datasets modeled by the operators $\hat{M}_{f_i}$

Part of that deformation may be to make the dataset descriptions more detailed

For the Wightman axioms, the Wigner-characteristic function is a generating function for the Wightman functions

Despite how simple the Wightman axioms look, there are no known well-defined interacting models in 1+3-dimensions, after 60+ years

How can we weaken the Wightman axioms?

- A Hilbert space $\mathcal{H}$ supports a unitary action of the Poincaré group
  there is a unique lowest energy Poincaré invariant vacuum vector $|v\rangle$
- Quantum fields are measurement operator–valued distributions, linear maps $\hat{M}: f \mapsto \hat{M}_f$ from a space of smearing functions into a $*$–algebra $\mathcal{A}$ of measurements
- Measurements support an action of the Poincaré group, $U(\Lambda)^\dagger \hat{M}_f U(\Lambda) = \hat{M}_{\Lambda(f)}$
- Microcausality: commutativity at space-like separation
- Completeness: the action of the $\hat{M}_f$ operators on $|v\rangle$ generates all of $\mathcal{H}$
- Time-slice axiom

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  Allow quantum fields to be *nonlinear* maps into $\mathcal{A}$
- Measurements support an action of the Poincaré group, $U(\Lambda)^\dagger \hat{M}_f U(\Lambda) = \hat{M}_{\Lambda(f)}$
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There are <u>*two*</u> linearities implicit in the Wightman axioms:

States are linear, $\rho(\lambda\hat{A}+\mu\hat{B}) = \lambda\rho(\hat{A})+\mu\rho(\hat{B})$, to ensure a probability interpretation

$\hat{M}_f$ is axiomatically linear in $f(x)$, $\hat{M}_f = \int \hat{M}(x) f(x) \mathrm{d}^4x$, $\hat{M}_{\lambda f+\mu g} = \lambda\hat{M}_f + \mu\hat{M}_g$

"A source fragmentation approach to interacting quantum field theory", arXiv:2109.04412

In signal analysis, with $f(x)$ and $g(x)$ as *window* or *modulation* functions
and (f, g) thought of as a *resonance* or as an *impulse–response pairing*,
linearity of the field is not *so* obvious that it should be *axiomatic*
We can use the Wightman axioms in linear *or* nonlinear forms

If we work with a more general construction, $\hat{M}_{Description_1}$, ..., $\hat{M}_{Description_n}$,
there is no reason to insist on linearity
For the Gaussian case, the matrix $\big(Description_i, Description_j\big)$
just has to be positive semi-definite — a structure known as a *Kernel*

This is not a *strong* argument,
because we might want to say that a *fundamental* field *should* be linear

In support, nonlinearity appears in real-space renormalization:

Suppose $f(x)$ is a *smearing function* on a square region,

which we split into $N=9$ fragments, then we apply *majority rule*,

-1	$+1$	-1
-1	$+1$	-1
-1	-1	$+1$

$$\longrightarrow -1, \quad \hat{M}_f^{[N]} = \epsilon\left[\sum_{j=1}^N \epsilon\left[\hat{M}_{f_j^{[N]}}\right]\right], \quad f = \sum_{j=1}^N f_j^{[N]}$$

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Iterated: $$\hat{M}_f^{[N_1,\ldots,N_k]}=\epsilon\Big[\textstyle\sum_{j=1}^{N_k} \epsilon\big[\hat{M}_{f_j^{[N_k]}}^{[N_1,\ldots,N_{k-1}]}\big]\Big],$$

$\hat{M}_f^{[\ldots]}$ is nonlinear in $f(x)$ and different from the inaccessible 'bare' $\hat{M}_f$,

but the 'dressed' $\hat{M}_f^{[\ldots]}$ is arguably still a "quantum field"

That's a slightly better argument that

we should allow and expect $\hat{M}_f$ to be nonlinear in $f(x)$

nonlinearity III — the renormalization group

Hollowood 2013: A measurement F is a function $F(\boldsymbol{\lambda}(\mu);\boldsymbol{\ell})_\mu = F(\boldsymbol{\lambda}(\mu');\boldsymbol{\ell})_{\mu'}, \quad \mu,\mu' > \boldsymbol{\ell}$

$\boldsymbol{\lambda}(\mu)$ are interaction parameters, $\boldsymbol{\ell}$ is a characteristic length scale, and μ,μ' are cutoffs

In more detail, write experimental results as functionals of test functions, $F[\boldsymbol{\lambda}(\mu); f_1, ..., f_n]_\mu$

Think of $f_1, ..., f_n$ as a complete description of an experimental apparatus entered into design software, enough to predict experimental results

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The cutoff scale appropriate for an experiment *also* depends on *details*,

$\mu,\mu' > \boldsymbol{\ell} \rightsquigarrow \mu = \mu[f_1, ..., f_n]$, which fixes an effective dynamics $\boldsymbol{\lambda}(\mu[f_1, ..., f_n])$

Hence, $\mathbf{F}[f_1, ..., f_n] = F[\boldsymbol{\lambda}(\mu[f_1, ..., f_n]); f_1, ..., f_n]_{\mu[f_1,...,f_n]}$

$\mathbf{F}[f_1, ..., f_n]$ is now a *more* nonlinear functional of $f_1, ..., f_n$

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Renormalization:

> If we use different cutoff scales, $\mu' = \mu'[f_1, \ldots, f_n]$, the experimental results
> $\mathbf{F}'[f_1, \ldots, f_n] = F[\boldsymbol{\lambda}(\mu'[f_1, \ldots, f_n]); f_1, \ldots, f_n]_{\mu'[f_1,\ldots,f_n]}$
> must be unchanged, $\mathbf{F}'[f_1, \ldots, f_n] = \mathbf{F}[f_1, \ldots, f_n]$

That's a more complicated argument, but the end result is the same:

we should allow and expect $\hat{M}_f$ to be nonlinear in $f(x)$

From Laura Ruetsche, "Locality in (Axiomatic) Quantum Field Theory", The Routledge Companion to Philosophy of Physics

For the Haag–Kastler axioms, *Strong Additivity* introduces linearity axiomatically as a precise way to rule out *holism*:

> Closely related to, but distinct from, various notions of locality is the idea of non-holism (see Healey (this volume) for detailed discussion of various notions of holism in physics). One way to express this notion is by the condition of strong additivity: the algebra $\mathfrak{N}(\cup_j \mathcal{O}_j)$ of observables associated with the union of any family of open bounded spacetime regions $\mathcal{O}_j$ coincides with the algebra $\vee_j \mathfrak{N}(\mathcal{O}_j)$ generated by the subalgebras $\mathfrak{N}(\mathcal{O}_j)$ associated with each of the individual regions – a precise way of framing in algebraic terms the idea that the whole is not greater than the sum of its parts.

Holism should not be excluded *axiomatically* by excluding nonlinearity, so, again, we should allow and expect $\hat{M}_f$ to be nonlinear in $f(x)$

Eastern Philosophy is often much more tolerant of holism than Western Philosophy is

Renormalization is a surreptitious way to inject nonlinearity

If $\hat{M}_f$ is nonlinear in f, we can construct multi-point operator-valued distributions,

$$\hat{M}(x_1, x_2) = \frac{\delta}{\delta f(x_1)} \frac{\delta}{\delta f(x_2)} \hat{M}_f \Big|_{f(x_1)=f(x_2)=0}$$

see Polyzou *et al* for this kind of structure used for bound states

If $\hat{M}_f$ is not expressible as a functional Taylor series (e.g. if we used $\hat{\phi}_{\tanh(f)}$),
test functions may wrap around singular points,
potentially resulting in discrete structure

nonlinearity V — multilinearity

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László&Tercsay, ClassQuantumGrav 2024, "On the running and the UV limit of Wilsonian renormalization group flows",
presents a complicated argument that
page 5: "the space of rescaled correlators" is "similar to that of n-variate distributions"
which becomes a surjective map
page 8: "for the special case of bosonic fields" on a "flat spacetime"

Statement (A): over generic spacetime manifolds, the space of rescaled correlators $z(C)^n \mathcal{G}_C^{(n)}$ ($C \in \{$coarse-grainings$\}$) of these flows form a topological vector space, which is Hausdorff, locally convex, complete, nuclear, semi-Montel and Schwartz. That is, they form a generalized function space having favorable properties similar to that of n-variate distributions.

The above theorem proves statement (A) in section 1. As seen, the topological vector space W_n has rather similar properties to the space of ordinary distributions $\mathcal{D}_n^{\times\prime}$. One may conjecture that $j[\mathcal{D}_n^{\times\prime}] \subset W_n$ saturates W_n. For the generic case, we were unable to construct a proof for this claim. However, for the special case of bosonic fields over affine spaces (flat spacetime), this surjectivity property is proved in the following section.

Given the reasoning on previous slides, let's just use nonlinearity

A conventional Lagrangian deformation $S[\hat{M}]$, from elementary textbooks, is applied to a free Wightman field,

$$\mathcal{Z}[\boldsymbol{j}] = \langle v|\, \hat{Z}_{\boldsymbol{j}}\, |v\rangle = \langle v|\, \mathsf{T}\Big[\mathrm{e}^{\mathrm{j}\hat{M}'_{\boldsymbol{j}}}\Big]\, |v\rangle = \frac{\langle v|\, \mathsf{T}\Big[\mathrm{e}^{\mathrm{j}S[\hat{M}]+\mathrm{j}\hat{M}_{\boldsymbol{j}}}\Big]\, |v\rangle}{\langle v|\, \mathsf{T}\Big[\mathrm{e}^{\mathrm{j}S[\hat{M}]}\Big]\, |v\rangle}$$

A Lorentz invariant generating functional $\mathcal{Z}[\boldsymbol{j}]$ is not well-defined

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A Lorentz invariant generating functional $\mathcal{Z}[\boldsymbol{j}]$ is not well-defined

The Reeh-Schlieder theorem asserts that for a free Wightman field *local operators acting on the vacuum vector $|v\rangle$ can approximate any vector*

$\longrightarrow$ for any regularization for which the vector $\hat{Z}_{\boldsymbol{j}}^{\text{regularized}}|v\rangle$ is of finite norm, we can approximate that vector by a vector $\hat{Z}^{\text{local,nonlinear}}[\hat{M}_{F_i[\boldsymbol{j}]}]|v\rangle$ that is constructed using only *nonlinear*, *local* functionals of $\boldsymbol{j}$

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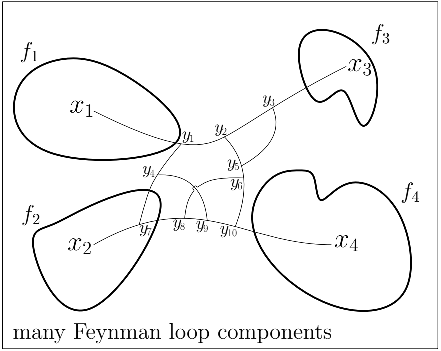
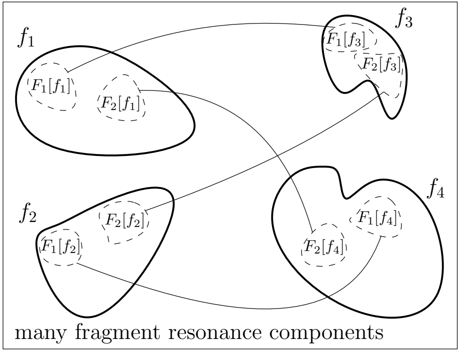
Renormalization introduces mass/length rescaling, which we can accommodate by using a collection of different fields $\hat{M}_f^{(i)}$

a reinvention of interacting QFT II

The physics $\approx \hat{Z}_{\boldsymbol{j}}^{\text{renormalized}(\mu)}|v\rangle \approx \hat{Z}^{\text{local,nonlinear}(\mu)}[\hat{M}^{(\text{i})}_{F_i[\boldsymbol{j}]}]|v\rangle$

$\langle v|\mathrm{T}\left[\hat{M}'_{f_4}\hat{M}'_{f_3}\hat{M}'_{f_2}\hat{M}'_{f_1}\right]|v\rangle$, *etc* can be generated by either $\longrightarrow$ sums of products of overlaps $(F_\alpha[f_i], F_\alpha[f_j])_\alpha$



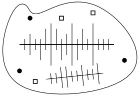


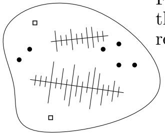
This is an inverse problem: find <u>*local*</u>, <u>*nonlinear*</u> *fragment functionals* $F_1[\cdot], F_2[\cdot], \ldots$
and measurement operators $\hat{M}^{(1)}_{F_1[\boldsymbol{j}]}, \hat{M}^{(2)}_{F_2[\boldsymbol{j}]}, \ldots$
that locally approximate what happens in the bulk for path integrals,
which works because of the Reeh-Schlieder theorem
The interplay between local and global is similar to holography

a reinvention of interacting QFT III

We can construct well-defined *Poincaré invariant fragment resonance theories*
for which, hopefully, $\hat{Z}^{\text{local,nonlinear}}[\hat{M}^{(i)}_{F_i[\boldsymbol{j}]}]|v\rangle \approx$ the physics

An analysis of renormalization suggests we *should* introduce nonlinearity
An analysis of path integrals shows that introducing nonlinearity is *enough*





For two regions, we can, *loosely*, think of there being correlations caused by noisy resonances between many pairs of antennas

In engineering terms, this is a nonlinear, generalized stochastic modulation-response formalism

We can construct well-defined *Poincaré invariant fragment resonance theories*
for which, hopefully, $\hat{Z}^{\text{local,nonlinear}}[\hat{M}^{(i)}_{F_i[\boldsymbol{j}]}]|v\rangle \approx$ the physics

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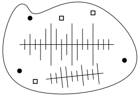

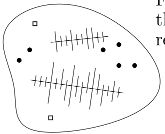

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$\rho'_v(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}'_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}'_{f_2}}\cdots) = \textit{some}$ manifestly Poincaré invariant functional of the λ_i and f_i ‡
We want to construct or decompose this functional in ways
that give us good intuitions about different experiments
or we might want to use much more elaborate mathematics

‡ This can be compared with "The Nonlinear Program"
but using only local functionals

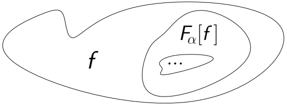
two tentative constructions

We can construct a recursive "cloud" of measurement operator *fragments*, within the support of f, as an iterated solution of an integral equation such as

$$\hat{M}_f = \hat{\phi}_f - \int \left[\hat{M}_{F_\alpha[f]}\right]^3 \mathrm{d}\mu(\alpha)$$

where $\mathrm{Supp}[F_\alpha[f]] \subseteq \mathrm{Supp}[f]$

for example f^2, $\tanh(f)$, $f{\cdot}(H \star f)$, ...



which integrates over different fragments $F_\alpha[\cdot]$ instead of over points in space-time

We can construct a nonlinear Gaussian field, as a first step, for which the 2-measurement VEV is nonlinear,

$$\langle v| \hat{M}_f^\dagger \hat{M}_g |v\rangle = (f, g) + \sum_\alpha (F_\alpha[f], F_\alpha[g])_\alpha,$$

using a sum of many binary relationships

instead of a sum over many paths

a concise list of the difficulties of QFT

from Oldofredi&Öttinger 2022:

1. Divergences (nonlinear fragment resonance theories, ...)
2. No precise ontological picture (signal analysis & classical measurement incompatibility)
3. No particles (nonlinearity & dispersion $\rightarrow$ solitons & caustics?)
4. Haag's theorem[‡] (nonlinearly constructed subalgebras)
5. The measurement problem (collapse as a joint probability construction)

[‡] Representations of the free field and of interacting fields are unitarily inequivalent
see Freeborn, Gilton, and Mitsch, "How Haag-tied is QFT, really?"

Given measurements $\hat{M}_{Description_1}$, ..., $\hat{M}_{Description_n}$,
all we need for us to be able to construct a Gaussian state
is *one* positive semi-definite $n{\times}n$ overlap matrix $\big(Description_i, Description_j\big)$

For interactions, we introduce and combine *many* such matrices, while ensuring
the properties required for $\rho_v(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}_{D_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}_{D_2}}\cdots)$ to be a state are satisfied

For QG, suppose we have a successful deformed Optics, $\rho'_v(\mathrm{e}^{\mathrm{j}\lambda_1\hat{M}'_{f_1}}\mathrm{e}^{\mathrm{j}\lambda_2\hat{M}'_{f_2}}\cdots)$,
which uses overlap matrices $(f_i, f_j)_\alpha$

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a bivector test function+information about a background: torsion | metric | non-metricity

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We can say how a measurement is done instead of saying what it measures

This gives a measurement theory perspective on 'quantizing gravity' or 'gravitizing quantum',
which doesn't have to be about 'gravitons'

towards Quantum Gravity II

In these days of LLMs, 'invariance' may be about more than diffeomorphisms

If we have two sets of *descriptions* of measurements for the same experiment,
$\mathcal{D}_1$ and $\mathcal{D}_2$, perhaps in different 'languages' and
with background modulations *described* in conceptually different ways,
then the predicted results should be the same

The coordinate systems we use for *Measurement Descriptions*
will be continuous because (whether the real world is continuous or not)
we can always *imagine* a measurement *between* previous measurements

When we *describe* measurements, is the metric connection still best?
When is torsion or non-metricity or something else also useful?

A *very* minimal axiom set for Quantum or QND Gravity:

1. Measurement operators $\hat{M}_{g_1}, \hat{M}_{g_2}, \ldots, \hat{M}_{g_N}$ are *nonlinear* in the *Measurement Descriptions* g_i
2. A *manifestly* Diffeomorphism invariant state $\longrightarrow$ GNS-construction of a Hilbert space

3a. QG: Spectrum Condition and Microcausality: $[\hat{M}_f, \hat{M}_g] = 0$ if f and g are causally separated

3b. QNDG: Universal measurement compatibility: $[\hat{\mathrm{M}}^{\mathrm{QND}}_f, \hat{\mathrm{M}}^{\mathrm{QND}}_g] = 0$ for all f and g

When discussing global properties, transformations, and classical measurement incompatibility, we can use a Poisson bracket, $\updownarrow$-operators, state projection operators, ...

3a. seems very difficult to get right on a curved space-time

3b. makes $\hbar$ *only* about the amplitude of a *deformed* Poincaré invariant noise