Flat BPS Domain Walls on $2d$ Kähler-Ricci Soliton

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arXiv:0901.0303v3 [hep-th] 26 Jun 2009

Abstract. In this paper we address several aspects of flat Bogomolnyi-Prasad-Sommerfeld (BPS) domain walls together with their Lorentz invariant vacua of $4d$ $N = 1$ supergravity coupled to a chiral multiplet. The scalar field spans a one-parameter family of $2d$ Kähler manifolds satisfying a Kähler-Ricci flow equation. We find that BPS equations and the scalar potential deform with respect to the real parameter related to the Kähler-Ricci soliton. In addition, the analysis using gradient and renormalization group flows are carried out to ensure the existence of Lorentz invariant vacua related to Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence.
Keywords: Domain Walls, Supergravity, Kähler-Ricci Flow.

1 Introduction

Deformation of $N = 1$ BPS supergravity domain walls on a Kähler-Ricci soliton, which generates a one-parameter family of Kähler manifolds, has been firstly studied in [1, 2] by us. In the future we plan to apply this work to study the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence in four dimensions and its evolution on a Kähler-Ricci soliton.

The aim of this paper is to consider a two dimensional model of $4d$ $N = 1$ supergravity BPS domain walls which generalizes a simple $\mathbb{C}\mathrm{P}^1$ model given in [1] as an example and the static case [3]. The $N = 1$ theory is coupled to a chiral multiplet where its complex scalar spans a one-parameter family of $2d$ Kähler manifolds generated by the $2d$ Kähler-Ricci flow equation [4][1]

$$\frac{\partial g_{z\bar{z}}}{\partial \tau} = -2R_{z\bar{z}}(\tau) = -2\,\partial_z\bar{\partial}_{\bar{z}}\ln g_{z\bar{z}}(\tau)\ , \tag{1.1}$$

whose solutions can be regarded as an area deformation of a Kähler surface for finite time. This can be easily seen, for example, when the initial geometry is a Kähler-Einstein surface

$$R_{z\bar{z}}(0) = \Lambda_2\, g_{z\bar{z}}(0)\ , \tag{1.2}$$

where $g_{z\bar{z}}(0) > 0$ and $\Lambda_2 > 0$. Then, in this case the solution of (1.1) is given by

$$g_{z\bar{z}}(\tau) = (1 - 2\Lambda_2\tau)\, g_{z\bar{z}}(0)\ , \tag{1.3}$$

and the area of the soliton is

$$|1 - 2\Lambda_2\tau|\sqrt{\det g(0)}\, dzd\bar{z}\ . \tag{1.4}$$

[1] See also [5, 6] for an excellent review of this subject.

Clearly, the soliton in (1.3) admits a singularity at $\tau = 1/2\Lambda_2$. For $0 \leq \tau < 1/2\Lambda_2$, the geometry is diffeomorphic to the initial geometry, while after hitting the singularity the new geometry has $\hat{g}_{z\bar{z}}(0) < 0$ and $\hat{\Lambda}_2 < 0$. This example shows that the $2d$ Kähler-Ricci soliton indeed defines a family of Kähler surfaces.

In this paper we consider a case where the walls preserve half of the supersymmetry of the parental theory described by a set of BPS equations. We use dynamical system analysis to study the BPS equations and the renormalization group (RG) flow equation to determine properties of the critical points describing Lorentz invariant vacua and their deformation.

Some results can be shortly mentioned as follows. The Kähler-Ricci soliton does play a role in determining the stability of the walls near critical points of the BPS equations describing Lorentz invariant vacua. As we will see, this occurs because the geometric (Kähler-Ricci) flow controls the signature of a Kähler metric. The analysis fails when the critical points become degenerate which might be a bifurcation point. Furthermore, the RG flow analysis shows that those vacua do not always exist particularly in the infrared (low energy scale) regions. It is important to note that the geometric flow may diverge at finite τ, where τ is a real parameter related to the Kähler-Ricci flow. In this situation, some quantities blow up.

The organization of this paper is as follows. In section 2 we give a quick introduction of $4d$ N=1 supergravity on $2d$ Kähler-Ricci soliton. Then we discuss some properties of flat BPS domain wall solutions of the theory in section 3. Section 4 provides a discussion about some aspects of Lorentz invariant vacua, gradient and renormalization group (RG) flows. We then apply the analysis to two simple models in section 5. We summarize the results in section 6.

2 $N = 1$ Chiral Supergravity on $2d$ Kähler-Ricci Soliton

In this section we review the four dimensional $N = 1$ supergravity coupled to a chiral multiplet in which the non-linear σ-model can be viewed as a one-parameter family of $2d$ Kähler manifolds generated by the Kähler-Ricci flow equation (1.1) and deformed with respect to the real parameter τ [1, 2]. The structure of this section follows rather closely reference [2].

The building blocks of the $N = 1$ theory are a gravitational multiplet and a chiral multiplet. The field components of the gravitational multiplet are a vierbein e^a_ν and a vector spinor ψ_ν where $a = 0, ..., 3$ and $\nu = 0, ..., 3$ are the flat and the curved indices, respectively. The chiral multiplet consists of a complex scalar z and a spin-$\frac{1}{2}$ fermion χ.

Then, a general $N = 1$ chiral supergravity Lagrangian together with its supersymmetry transformation can be constructed. A pedagogical review of this subject can be found, for example, in [7]. Here, we assemble the terms which are useful for our analysis. The

bosonic part of the $N = 1$ supergravity Lagrangian has the form

$$\mathcal{L}^{N=1} = -\frac{M_P^2}{2} R + g_{z\bar{z}}(z, \bar{z}; \tau)\, \partial_\nu z\, \partial^\nu \bar{z} - V(z, \bar{z}; \tau) \ , \tag{2.1}$$

where M_P is the Planck mass [2], R is the Ricci scalar of the four dimensional spacetime; and the pair $(z, \bar{z})$ spans a Hodge-Kähler manifold with metric $g_{z\bar{z}}(z, \bar{z}; \tau) \equiv \partial_z \partial_{\bar{z}} K(z, \bar{z}; \tau)$; and $K(z, \bar{z}; \tau)$ is a real function, called the Kähler potential. The $N = 1$ scalar potential $V(z, \bar{z}; \tau)$ has the form

$$V(z, \bar{z}; \tau) = e^{K(\tau)/M_P^2} \left(g^{z\bar{z}}(\tau) \nabla_z W\, \bar{\nabla}_{\bar{z}} \bar{W} - \frac{3}{M_P^2} W \bar{W} \right) \ , \tag{2.2}$$

where W is a holomorphic superpotential and $\nabla_z W \equiv (dW/dz) + (K_z(\tau)/M_P^2) W$. The supersymmetric invariance of the Lagrangian (2.1) is guaranteed by the following transformations of the fields up to three-fermion terms [3]

$$\begin{aligned}
\delta\psi_{1\nu} &= M_P \left(D_\nu \epsilon_1 + \frac{\mathrm{i}}{2} e^{K(\tau)/2M_P^2}\, W \gamma_\nu \epsilon^1 + \frac{\mathrm{i}}{2M_P} Q_\nu(\tau) \epsilon_1 \right) \ , \\
\delta\chi^z &= \mathrm{i} \partial_\nu z\, \gamma^\nu \epsilon^1 + N^z(\tau) \epsilon_1 \quad , \\
\delta e_\nu^a &= -\frac{\mathrm{i}}{M_P} \left(\bar{\psi}_{1\nu}\, \gamma^a \epsilon^1 + \bar{\psi}_\nu^1\, \gamma^a \epsilon_1 \right) , \\
\delta z &= \bar{\chi}^z \epsilon_1 \ ,
\end{aligned} \tag{2.3}$$

where $N^z(\tau) \equiv e^{K(\tau)/2M_P^2}\, g^{z\bar{z}}(\tau) \bar{\nabla}_{\bar{z}} \bar{W}$, $g^{z\bar{z}}(\tau) = (g_{z\bar{z}}(\tau))^{-1}$, and the $U(1)$ connection $Q_\nu(\tau) \equiv -\left(K_z(\tau)\, \partial_\nu z - K_{\bar{z}}(\tau)\, \partial_\nu \bar{z} \right)$. Here, we have also introduced $\epsilon_1 \equiv \epsilon_1(x, \tau)$. One can then show that using (1.1), the evolution equations of both the scalar potential (2.2) and the shifting quantity N^z can be written as [1]

$$\begin{aligned}
\frac{\partial N^z(\tau)}{\partial \tau} &= 2R^z_{\ z}(\tau) N^z(\tau) + \frac{K_\tau(\tau)}{2M_P^2} N^z(\tau) + g^{z\bar{z}}(\tau) \frac{K_{\bar{z}\tau}(\tau)}{M_P^2} e^{K(\tau)/2M_P^2}\, \bar{W} \ , \\
\frac{\partial V(\tau)}{\partial \tau} &= \frac{\partial N^z(\tau)}{\partial \tau} N_z(\tau) + \frac{\partial N_z(\tau)}{\partial \tau} N^z(\tau) - \frac{3K_\tau(\tau)}{M_P^2} e^{K(\tau)/M_P^2} |W|^2 \ ,
\end{aligned}$$

where $R^z_{\ z} \equiv g^{z\bar{z}} R_{z\bar{z}}$.

It is important to remark, as mentioned in the previous section, that the metric $g_{z\bar{z}}(\tau)$ could possibly turn into a negative definite metric. Such a case would yield a non-unitary theory with ghosts. At the classical level, it may be possible to formally write down theories of this type. However, at the quantum level this would lead to negative norm states in the standard Fock space which are discarded as unphysical [4].

3 Flat BPS Domain Walls

Now we give a discussion about the ground states which partially break Lorentz invariance, *i.e.*, domain walls. The first step is to take the ansatz metric as

$$ds^2 = a^2(u, \tau)\, \eta_{\underline{\lambda}\,\underline{\nu}}\, dx^{\underline{\lambda}} dx^{\underline{\nu}} - du^2 \quad , \tag{3.1}$$

[2] Setting $M_P \to +\infty$, one gets the global $N = 1$ chiral supersymmetric theory.

[3] The symbol D_ν here is different with the one in reference [3]. D_ν here is defined as $D_\nu \equiv \partial_\nu - \frac{1}{4} \gamma_{ab}\, \omega_\nu^{ab}$.

[4] We thank M. Zagermann for pointing out these aspects.

where $\underline{\lambda}$, $\underline{\nu} = 0, 1, 2$, $a(u, \tau)$ is the warped factor, and $\eta_{\underline{\lambda}\underline{\nu}}$ is the flat Minkowskian metric. Writing the supersymmetry transformation (2.3) on the background (3.1) and setting $\psi_{1\mu} = \chi^z = 0$, then a set of equations that partially preserves supersymmetry can be obtained in the following

$$\begin{aligned} \frac{a'}{a} &= \pm\, \mathcal{W}(\tau) \,, \\ z' &= \mp 2 g^{z\bar{z}}(\tau) \bar{\partial}_{\bar{z}} \mathcal{W}(\tau) \,, \\ \bar{z}' &= \mp 2 g^{z\bar{z}}(\tau) \partial_z \mathcal{W}(\tau) \,, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} a' &\equiv \partial a / \partial u \,, \\ \mathcal{W}(\tau) &\equiv e^{K(\tau)/M_P^2} \, |W| \,, \end{aligned} \tag{3.3}$$

and we have assumed $\epsilon_1(u, \tau)$ and $z(u, \tau)$. The last two equations in (3.2) are called BPS equations. Moreover, in the conformal field theory (CFT) picture one can then define a beta function for all scalar fields as

$$\beta \equiv a \frac{\partial z}{\partial a} = -2 g^{z\bar{z}}(\tau) \frac{\bar{\partial}_{\bar{z}} \mathcal{W}}{\mathcal{W}} \quad , \tag{3.4}$$

after using (3.2), together with its complex conjugate describing another supersymmetric RG flows. In this picture the scalars can be viewed as coupling constants and the warp factor a is playing the role of an energy scale [8, 9]. It is worth mentioning that equation (3.2) can also be derived in more general four dimensional gravity theory without introducing supersymmetry (2.3). In this case one has to solve the Einstein's field equations together with the equation of motions of the complex scalar fields. These equations can be constructed by varying the Lagrangian (2.1) with respect to the spacetime metric $g_{\mu\nu}$ and the scalar field z, and also, applying the necessary condition that the variation of τ vanishes. As a result, we find that the real function $\mathcal{W}(\tau)$ has a general form and

$$\mathcal{W}(\tau) \neq e^{K(\tau)/M_P^2} \, |W| \,. \tag{3.5}$$

The corresponding scalar potential is given by

$$V(z, \bar{z}; \tau) = 4\, g^{z\bar{z}}(\tau)\, \partial_z \mathcal{W}(\tau)\, \bar{\partial}_{\bar{z}} \mathcal{W}(\tau) - \frac{3}{M_P^2}\, \mathcal{W}^2(\tau) \,. \tag{3.6}$$

Then, the CFT picture can also be defined by the beta function (3.4) in a more general way, namely equation (3.5) holds.

Finally, it is straightforward to show that the critical points of the BPS equations in (3.2) are related to the following condition

$$\partial_z \mathcal{W} = \bar{\partial}_{\bar{z}} \mathcal{W} = 0 \quad , \tag{3.7}$$

which implies that

$$\partial_z V = \bar{\partial}_{\bar{z}} V = 0 \quad . \tag{3.8}$$

This means that the critical points of $\mathcal{W}(z, \bar{z})$ are somehow related to the critical points of the $N = 1$ scalar potential $V(z, \bar{z})$. Moreover, in the view of (3.4), these points are in ultraviolet (UV) region if $a \to \infty$ and in infrared (IR) region if $a \to 0$. Thus, the RG flow interpolates between UV and IR critical points.

4 Supersymmetric Vacua and AdS/CFT Correspondence

Let us begin our discussion by mentioning that from (3.7) a critical point of the real function $\mathcal{W}(\tau)$, say p_0, is in general $p_0 \equiv (z_0(\tau), \bar{z}_0(\tau))$ due to the geometric flow (1.1). Such a point exists in the asymptotic regions, namely around $u \to \pm\infty$. The form of the scalar potential (2.2) at p_0 is

$$V(p_0;\tau) = -\frac{3}{M_P^2}\,\mathcal{W}^2(p_0;\tau) \equiv -\frac{3}{M_P^2}\,\mathcal{W}_0^2\ . \tag{4.1}$$

For the case of the flat domain walls discussed in [3] equation (4.1) can be viewed as the cosmological constant of the spacetime at the vacuum. So the spacetime is AdS with negative cosmological constant for $\mathcal{W}_0 \neq 0$. Here, we exclude the case of Minkowskian vacua, where $\mathcal{W}_0 = 0$.

The next step is to consider some aspects of the critical points of $\mathcal{W}(\tau)$ and the vacua of the theory defined by the critical points of the scalar potential (2.2). First, we write down the eigenvalues of $\mathcal{W}(\tau)$ and the scalar potential in (2.2) as

$$\begin{aligned}
\lambda^{\mathcal{W}}_{1,2}(\tau) &= \frac{g_{z\bar{z}}(p_0;\tau)}{M_P^2}\mathcal{W}_0 \pm 2|\partial_z^2\mathcal{W}_0|\ , \\
\lambda^{V}_{1,2}(\tau) &= -4\left(\frac{g_{z\bar{z}}(p_0;\tau)}{M_P^4}\mathcal{W}_0^2 - 2g^{z\bar{z}}(p_0;\tau)|\partial_z^2\mathcal{W}_0|^2\right) \\
&\quad \pm 4\frac{\mathcal{W}_0}{M_P^2}|\partial_z^2\mathcal{W}_0|\ ,
\end{aligned} \tag{4.2}$$

respectively, where

$$\partial_z^2\mathcal{W}_0 \equiv \frac{e^{K(p_0;\tau)/M_P^2}\,\bar{W}(\bar{z}_0)}{2\mathcal{W}(p_0;\tau)}\left(\frac{d^2W}{dz^2}(z_0) + \frac{K_{zz}(p_0;\tau)}{M_P^2}W(z_0) + \frac{K_z(p_0;\tau)}{M_P^2}\frac{dW}{dz}(z_0)\right)\ . \tag{4.3}$$

The analysis of the above eigenvalues has been carried out in [2, 3]. Here, we just summarize it in the following table.

Table 1 Deformation of Vacua due to a Kähler-Ricci Soliton

Kähler Metric	Condition	Critical Points of $\mathcal{W}(\tau)$	Type of Vacua
$g_{z\bar{z}}(\tau) > 0$	$\lvert\partial_z^2\mathcal{W}_0\rvert > \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	saddle	local minimum saddle degenerate
	$\lvert\partial_z^2\mathcal{W}_0\rvert < \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	local minimum	local maximum
	$\lvert\partial_z^2\mathcal{W}_0\rvert = \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	degenerate	intrinsic degenerate
$g_{z\bar{z}}(\tau) < 0$	$\lvert\partial_z^2\mathcal{W}_0\rvert > \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	saddle	local maximum saddle degenerate
	$\lvert\partial_z^2\mathcal{W}_0\rvert < \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	local maximum	local minimum
	$\lvert\partial_z^2\mathcal{W}_0\rvert = \frac{g_{z\bar{z}}(p_0;\tau)}{2M_P^2}\mathcal{W}_0$	degenerate	intrinsic degenerate

As we saw in the table, it is possible to have a parity transformation of the vacua caused by the Kähler-Ricci soliton. This can be easily seen in the local minimum vacua in which they are changing to a local maximum vacua as the metric deformed to be a negative definite metric.

Furthermore, we have to check the stability of the walls near p_0 by expanding the BPS equations in (3.2) near p_0 to first order. It turns out that the first order expansion matrix has the eigenvalues

$$\Lambda_{1,2} = \mp\frac{\mathcal{W}_0}{M_P^2} - 2g^{z\bar{z}}(p_0;\tau)|\partial_z^2\mathcal{W}_0|\,, \tag{4.4}$$

which describe the gradient flows. If $g^{z\bar{z}}(p_0;\tau)$ is positive definite for finite τ, then the stable flow is a stable node flowing along local minimum vacua and the stable curve of the saddle vacua, whereas the unstable flow is a saddle flowing along the local maximum vacua. If $g^{z\bar{z}}(p_0;\tau)$ becomes negative definite for finite τ, then we have an unstable node flowing along the local maximum vacua and the unstable direction of the saddle vacua, while the stable saddle flow is flowing along the local minimum vacua. These show that in the latter case the walls are mostly unstable, and they are stable if the gradient flow is the stable saddle flow. In addition, Our linear analysis fails if one eigenvalue in (4.4) is zero which is possibly a fold bifurcation point. This happens at degenerate critical points of the real function which correspondingly, are related to the degenerate critical points of the scalar potential (2.2).

Now we turn to consider the RG flows. As we have mentioned in the preceding section, this function can also be used to determine the nature of the critical point p_0, namely, it can be interpreted as the UV or IR limit in the CFT picture. To begin, we expand the beta function (3.4) to first order around p_0 and then we have a matrix whose eigenvalues are

$$\lambda^{\mathcal{U}}_{1,2} = \frac{1}{M_P^2} \pm \frac{2g^{z\bar{z}}(p_0;\tau)}{\mathcal{W}_0}\,|\partial_z^2\mathcal{W}_0|\ . \tag{4.5}$$

Let us choose a model where the UV region is $u \to +\infty$ as $a \to +\infty$, while the IR region is $u \to -\infty$ as $a \to 0$. In the UV region at least one of the eigenvalues (12) should be positive. Moreover, since it is possible to have zero and a negative eigenvalue, i.e. for minus sign, as the parameters in $\mathcal{W}(\tau)$ or the flow parameter τ vary, the RG flow fails to depart the UV region in this direction. So, the flow is stable along the positive eigenvalue and we have all possible vacua in the UV region.

On the other hand, in the IR region the RG flow approaches a critical point in the direction of negative eigenvalue. However, since this eigenvalue could be zero and positive as the parameters vary, then the IR critical points would vanish. In the scalar potential picture there are no IR local maximum vacua for the positive definite Kähler metric, *i.e.* $g_{z\bar{z}}(\tau) > 0$, while no IR local minimum vacua for the negative definite Kähler metric, *i.e.* $g_{z\bar{z}}(\tau) < 0$. In addition, for both cases intrinsic degenerate vacua do not exist in this IR region.

It is important to note that the above analysis fail if there exists a singularity of the Kähler-Ricci soliton at a particular value of τ.

5 Two $\mathbb{C}\mathrm{P}^1(p_0)$ Models

In this section we organize the discussions into two parts. The first part is dealing with a linear superpotential, while the second part is a model with harmonic superpotential. Moreover, this section is devoted for considering a complex projective manifold with the center $p_0 \equiv (z_0, \bar{z}_0) \neq 0$ is denoted by $\mathbb{C}\mathrm{P}^1(p_0)$ and the Kähler potential is given by

$$K(0) = \ln(1 + |z - z_0|^2)\,, \tag{5.1}$$

where p_0 is a vacuum and for $p_0 = 0$ we just write $\mathbb{C}\mathrm{P}^1$. Then the τ-dependent Kähler potential is given by

$$K(\tau) = \sigma(\tau)\ln(1 + |z - z_0|^2)\,, \tag{5.2}$$

where $\sigma(\tau) \equiv (1-4\tau)$. For $\tau < 1/4$ we have a manifold which **is** diffeomorphic to $\mathbb{C}\mathrm{P}^1(p_0)$, while after hitting the singularity at $\tau = 1/4$ we have $\widehat{\mathbb{C}\mathrm{P}^1}(p_0)$ which is called a parity manifold of $\mathbb{C}\mathrm{P}^1(p_0)$.

5.1 A Model with Linear Superpotential

In this subsection we consider a model with linear superpotential

$$W(z) = a_0 + a_1\,(z - z_0)\,, \tag{5.3}$$

where $a_0, a_1 \in \mathbb{R}$. Equation (3.7) in this case becomes

$$a_1 + \frac{\sigma(\tau)(\bar{z} - \bar{z}_0)}{M_P^2(1 + |z - z_0|^2)}(a_0 + a_1\,(z - z_0)) = 0\,. \tag{5.4}$$

Taking $z = z_0$, it follows that $a_1 = 0$ and the ground state is an unstable isolated AdS spacetime with the cosmological constant

$$V(p_0;\tau) = -\frac{3|a_0|^2}{M_P^2}\,, \tag{5.5}$$

for $a_0 \neq 0$ and describing unstable walls for $\tau < 1/4$. The ground state becomes stable for $\tau > 1/4$.

If there exists another vacuum $p_0' \neq p_0$, by defining $z_0' \equiv x_0' + \mathrm{i}y_0'$ with $x_0', y_0' \in \mathbb{R}$, then the condition (5.4) gives

$$\begin{aligned} x_0'(\tau) &= \frac{1}{2(\sigma(\tau) + M_P^2)}\left[-\frac{\sigma(\tau)a_0}{a_1} \pm \left(\left(\frac{\sigma(\tau)a_0}{a_1}\right)^2 - 4M_P^2(\sigma(\tau) + M_P^2)\right)^{1/2}\right] + x_0\,, \\ y_0'(\tau) &= y_0\,, \end{aligned} \tag{5.6}$$

with $a_1 \neq 0$. Moreover, since $x_0', y_0' \in \mathbb{R}$ the flow parameter τ must be defined in the following interval [1]

$$\begin{aligned} \tau &\leq \frac{1}{4} - \frac{M_P^2}{2}\left(\frac{a_1}{a_0}\right)^2\left[1 + \left(1 + \left(\frac{a_0}{a_1}\right)^2\right)^{1/2}\right], \\ \tau &\geq \frac{1}{4} + \frac{M_P^2}{2}\left(\frac{a_1}{a_0}\right)^2\left[-1 + \left(1 + \left(\frac{a_0}{a_1}\right)^2\right)^{1/2}\right], \end{aligned} \tag{5.7}$$

before and after the singularity at $\tau = 1/4$, respectively. For the shake of simplicity we take $a_0 \gg a_1 > 0$. Afterward, we have the same discussion as in [1]. Here we summarized the results in the following table.

Table 2 Deformation of Vacua in a $\mathbb{C}\mathrm{P}^1(p_0)$ Model with Linear Superpotential

Kähler Metric	Interval	Type of Vacua	Region	Stability
$g_{z\bar{z}}(p_0;\tau) > 0$ $0 \le \tau < 1/4$	$\tau < -\frac{a_0 M_P^2}{8a_1}$	local maximum	UV	unstable
	$-\frac{a_0 M_P^2}{8a_1} < \tau < -\frac{a_0 M_P^2}{8a_1\sqrt{2}}$	saddle	UV, IR	stable
	$-\frac{a_0 M_P^2}{8a_1\sqrt{2}} < \tau \lesssim \frac{1}{4} - \frac{a_1 M_P^2}{2a_0}$	local minimum	UV, IR	stable
	$\tau \approx -\frac{a_0 M_P^2}{8a_1\sqrt{2}}$	degenerate	UV, IR	stable
	$\tau \approx -\frac{a_0 M_P^2}{8a_1}$	intrinsic degenerate	UV	undetermined
$g_{z\bar{z}}(p_0;\tau) < 0$ $\tau > 1/4$	$\frac{1}{4} + \frac{a_1 M_P^2}{2a_0} \lesssim \tau < \frac{a_0 M_P^2}{8a_1\sqrt{2}}$	local maximum	UV, IR	unstable
	$\frac{a_0 M_P^2}{8a_1\sqrt{2}} < \tau < \frac{a_0 M_P^2}{8a_1}$	saddle	UV, IR	unstable
	$\tau > \frac{a_0 M_P^2}{8a_1}$	local minimum	UV	stable
	$\tau \approx \frac{a_0 M_P^2}{8a_1\sqrt{2}}$	degenerate	UV, IR	unstable
	$\tau \approx \frac{a_0 M_P^2}{8a_1}$	intrinsic degenerate	UV	undetermined

Our comments are in order. As we have seen in the above table, all possible cases may exist in the UV region. However in the IR region, this is not the case. Before the singularity at $\tau = 1/4$ the local maximum vacua do not exist in the IR region, while after the singularity, the local minimum vacua are forbidden. In both cases, there are no intrinsic degenerate vacua in this IR region.

5.2 A Model with Harmonic Superpotential

In this subsection we discuss properties of a model with the harmonic superpotential

$$W(z) = A_0 \sin(kz)\,, \tag{5.8}$$

where A_0 and k are real [3]. Similar to the preceding subsection, taking $z = z_0$, the condition (3.7) results in

$$y_0 = 0\,,\ x_0 = (n + \frac{1}{2})\frac{\pi}{k}\,, \quad n = 0, \pm 1, \pm 2, ..., \tag{5.9}$$

where we have also defined $z_0 \equiv x_0 + \mathrm{i}y_0$. For the case at hand, the ground state is an AdS spacetime for $A_0 \neq 0$ with

$$V(p_0;\tau) = -\frac{3|A_0|^2}{M_P^2}\,. \tag{5.10}$$

One can further obtain

$$\begin{aligned} \lambda^{\mathcal{W}}_{1,2} &= \left(\frac{\sigma(\tau)}{M_P^2} \pm k^2\right)|A_0|\,, \\ \lambda^{V}_{1,2} &= 2|A_0|^2\left(\sigma(\tau)^{-1}\,k^4 - \frac{2\sigma(\tau)}{M_P^4} \pm \frac{k^2}{M_P^2}\right)\,, \end{aligned} \tag{5.11}$$

from (4.2) and in this model the possible bifurcation points occur at $k = \pm|\sigma(\tau)|^{1/2}/M_P$ and $\tau \neq 1/4$.

Finally we want to mention that if there exists another vacuum $z_0' \neq z_0$, we will get an implicit equation which is complicated to have its exact solution. This aspect will be discussed elsewhere.

6 Conclusions

We have studied general properties of the BPS domain walls on a $2d$ Kähler-Ricci soliton. The critical points of the scalar potential describing the supersymmetric vacua do deform with respect to the real parameter τ related to the geometric soliton.

Firstly, for $g_{z\bar{z}}(\tau) > 0$ with finite τ, the real function $\mathcal{W}(\tau)$ have three types of critical points, namely the local minima, the saddles, and the degenerate critical points. The local minima correspond to the local maximum vacua, whereas the saddles are mapped into a local minimum or saddle vacua. The degenerate critical points imply intrinsic degenerate vacua. On the other side, if the metric $g_{z\bar{z}}(\tau)$ becomes negative definite for finite τ, then the local minima of $\mathcal{W}(\tau)$ turn into a local maxima. At the level of vacua such situation could possibly occur. This shows the existence of the parity transformation for the critical points or the vacua of the theory.

The analysis using gradient flows shows that if the metric $g_{z\bar{z}}(\tau) > 0$ for finite τ, then we have stable nodes flowing along the local minimum vacua and the stable curve of the saddle vacua. The other flows are unstable saddles flowing along local maximum vacua. Next, if $g_{z\bar{z}}(\tau) < 0$ for finite τ, then there are unstable nodes flowing along the local maximum and the unstable direction of the saddle vacua, whereas the stable saddle flows are flowing along the local minimum vacua. These facts show that as mentioned above, there exists a possibility of having a parity transformation of gradient flows caused by the soliton. In addition, the analysis is failed when the critical points of $\mathcal{W}(\tau)$ become degenerate which is the bifurcation point of the gradient flows.

On the other side, the RG flows determine whether such gradient flows exist in the UV and IR regions. In the UV region all possible vacua are allowed for all value of $g_{z\bar{z}}(\tau)$. However, such situation does not valid in the IR regions. In the case where $g_{z\bar{z}}(\tau) > 0$, we do not have IR local maximum vacua, whereas for the case $g_{z\bar{z}}(\tau) < 0$ there are no IR local minimum vacua.

At the end we applied the general analysis to two simple models on $\mathbb{C}\mathrm{P}^1(p_0)$. If we take $z = z_0$, then the cosmological constants do not depend on τ. However, the second order analysis does depend on τ which shows that the properties of the ground states may be changed because of the Kähler-Ricci soliton. If there is another vacuum $z_0' \neq z_0$, then the case becomes complicated. Thus, for example it is easier for the case of linear superpotential.

Acknowledgment

We thank L. Andrianopoli, H. Alatas, Arianto, A. N. Atmaja, K. Yamamoto, J. M. Maldacena, T. Mohaupt, and M. Zagermann for useful discussions. One of us (BEG) is grateful to the organizer of the Abdus Salam ICTP Spring School on Superstring Theory and Related Topics for providing a good atmosphere where parts of this work was done. We finally thank M. Satriawan for careful reading the manuscript and correcting the grammar. This work was initiated by Riset KK ITB 2008 under contract No. 035/K01.7/PL/2008 and is extensively supported by Hibah Kompetensi DIKTI 2008 No. 040/HIKOM/DP2M/2008 and Hibah Kompetensi DIKTI 2009 No. 223/SP2H/PP/DP2M/V/2009.

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