$$\mathcal{L}=\sqrt{H(U)}\sqrt{1+\beta(U)U_x^2}$$

$$\frac{\partial\mathcal{L}}{\partial U_x}=\sqrt{H}\frac{\beta U_x}{\sqrt{1+\beta U_x^2}}\Rightarrow\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial U_x}=\frac{H_U U_x}{2\sqrt{H}}\frac{\beta U_x}{\sqrt{1+\beta U_x^2}}+\sqrt{H}\frac{\beta_U U_x^2+\beta U_{xx}}{\sqrt{1+\beta U_x^2}}-\sqrt{H}\frac{\beta U_x\left(\beta_U U_x^3+2\beta U_x U_{xx}\right)}{2\left(1+\beta U_x^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial\mathcal{L}}{\partial U}=\frac{H_U}{2\sqrt{H}}\sqrt{1+\beta U_x^2}+\sqrt{H}\frac{\beta_U U_x^2}{2\sqrt{1+\beta U_x^2}}$$

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial U_x}=\frac{\partial\mathcal{L}}{\partial U}\Rightarrow$$

$$\Rightarrow\frac{H_U U_x}{2\sqrt{H}}\frac{\beta U_x}{\sqrt{1+\beta U_x^2}}+\sqrt{H}\frac{\beta_U U_x^2+\beta U_{xx}}{\sqrt{1+\beta U_x^2}}-\sqrt{H}\frac{\beta U_x\left(\beta_U U_x^3+2\beta U_x U_{xx}\right)}{2\left(1+\beta U_x^2\right)^{\frac{3}{2}}}=\frac{H_U}{2\sqrt{H}}\sqrt{1+\beta U_x^2}+\sqrt{H}\frac{\beta_U U_x^2}{2\sqrt{1+\beta U_x^2}}\Rightarrow$$

$$\Rightarrow\frac{H_U\beta U_x^2}{H}\left(1+\beta U_x^2\right)+2\left(\beta_U U_x^2+\beta U_{xx}\right)\left(1+\beta U_x^2\right)-\beta U_x^2\left(\beta_U U_x^2+2\beta U_{xx}\right)=\frac{H_U}{H}\left(1+\beta U_x^2\right)^2+\beta_U U_x^2\left(1+\beta U_x^2\right)\Rightarrow$$

$$\Rightarrow-\frac{H_U}{H}\left(1+\beta U_x^2\right)+\beta_U U_x^2+2\beta U_{xx}=0\xrightarrow{S\equiv U_x\Rightarrow U_{xx}=SS_U}-\frac{H_U}{H}\left(1+\beta S^2\right)+\beta_U S^2+2\beta SS_U=0\Rightarrow$$

$$\Rightarrow S_U+\frac{1}{2}\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)S=\frac{H_U}{2\beta HS}\xrightarrow{Z\equiv S^2\Rightarrow S_U=\frac{Z_U}{2\sqrt{Z}}}\frac{Z_U}{2\sqrt{Z}}+\frac{1}{2}\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)\sqrt{Z}=\frac{H_U}{2\beta H\sqrt{Z}}\Rightarrow$$

$$\Rightarrow Z_U+\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)Z=\frac{H_U}{\beta H}\Rightarrow YZ_U+Y\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)Z=Y\frac{H_U}{\beta H}\Rightarrow$$

$$Y_U=Y\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)\Rightarrow\frac{Y_U}{Y}=\left(\frac{\beta_U}{\beta}-\frac{H_U}{H}\right)\Rightarrow\ln Y=\ln\beta-\ln H\Rightarrow Y=\frac{\beta}{H}$$

$$\Rightarrow\frac{\beta}{H}Z_U+\left(\frac{\beta_U}{H}-\frac{\beta H_U}{H^2}\right)Z=\frac{H_U}{H^2}\Rightarrow\left(\frac{\beta}{H}Z\right)_U=\left(-\frac{1}{H}\right)_U\Rightarrow Z=-\frac{1}{\beta}\Rightarrow S=\sqrt{-\frac{1}{\beta}}\Rightarrow U_x=\sqrt{-\frac{1}{\beta}}$$