Cantor Diagonal Argument-false

Richard L. Hudson 8-4-2021

abstract

This analysis shows Cantor's diagonal argument published in 1891 cannot form a new sequence that is not a member of a complete list. The proof is based on the pairing of complementary sequences forming a binary tree model.

1. the argument

Assume a complete list L of random infinite sequences. Each sequence S is a unique infinite pattern of symbols (0 or 1). A sample of a random list begins as:

S_1 100101...
S_2 010011...
S_3 110011...
S_4 100000...
S_5 000111...
S_6 111001...

A sequence p is formed from the diagonal elements (underlined) by applying the rule, if 0 then 1 else 0, to each position from left to right. The diagonal d=110011... is transformed via the substitution rule to the horizontal p = 001100...

1.1 Cantor's conclusion

Since p differs from each S in the sample by construction, it will differ from all S in the list L, therefore a new sequence p will be formed not in the list L. The set of integers N is not sufficient to count the list L. [1]

2. binary tree

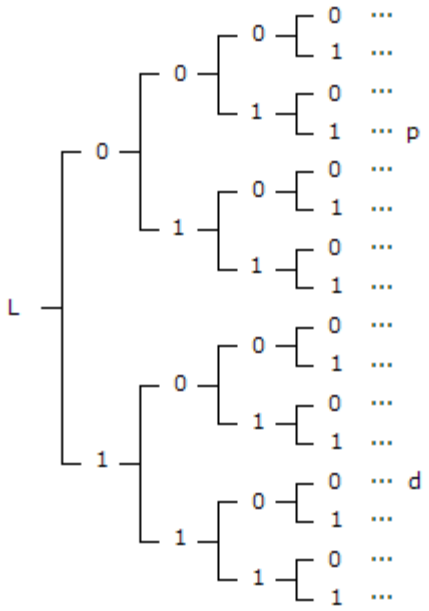

fig.1

The binary tree (fig.1) shows the beginning of all possible sequences, each corresponding to a unique linear path from left to right. All S must begin with 0 or 1, thus all would be contained in the tree if extended without limit. The tree therefore is a representation of L as defined in sec.1. Sequence p and its complement d are included in L as noted in fig.1. The tree is symmetrical relative to a horizontal line through L. If the tree is rotated 180° on the line, the symbols 0 and 1 are interchanged showing the pair of sequences d and p are complementary and mirror images. The number of paths at each position k equals 2^k.

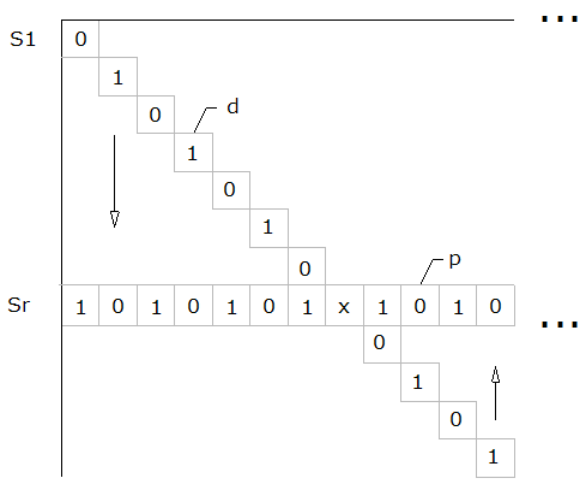

fig. 2

Fig.2 shows how p, the sequence formed from the diagonal d is actually an element of L, yet goes undetected. Sequence d is a simple alternating symbol beginning with 0.
The complement p is therefore an alternating symbol beginning with 1. The arrows show the correspondence forward and backward. Sr can be anywhere in L.
The position x is 1 for d and 0 for p. Sr is different from d at x but would have to be compared in all positions to reveal itself as the complement of d.

3. the error

Cantor's conclusion in 1.1 is false.

S_1 100101... 3
S_2 010011... 1
S_3 110011... 0
S_4 100000... 3
S_5 000111... 3
S_6 111001... 2

A random selection S_3 is compared to each S in the sample for quantity of differences (column 3). For a complete set L, the sequence S_3 differs from all S in the sample except one, itself. If S_3 is excluded from L by classifying it as new without further verification, then it will differ from all remaining S in the sample. That status would be true of all remaining members of L. Those S can't be new and existing members of L, a contradiction.

conclusion

1. Each unique sequence S must differ from all other S in the list by at least one position, the greatest difference being all positions for S and its complement S', d and p in the example. Any S cannot differ from itself.
2. Sequences S are independent of orientation, horizontal, vertical, or diagonal, a fact ignored by Cantor. The diagonal d=110011... is already in the list as line 3, sec.1. Since his method only makes one comparison per sequence, it does not provide a means of detecting the complementary sequence, or that d is a duplication.
3. If d is not new, then neither is p, since they occur in complementary pairs.
4. If the unlimited sequences S_1, S_2, S_3, S_4,..., could be produced, a corresponding integer could be assigned from N, with the assurance that N is inexhaustible.

references

1. Cantor's Diagonal Argument, Wikipedia, Mar 2015
2. THE LOGIC MUSEUM