Mathematical Challenges

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1. Find an integer with ten digits, such that the first n digits ($1 \leq n \leq 10$) are divisible by n.

   Reason: Easy test of divisibility properties.

   Solution:

   - An integer is divisible by 1 without remainder.
   - An integer is divisible by 2 without remainder if the last digit is even.
   - An integer is divisible by 3 without remainder if its checksum is divisible by 3.
   - An integer is divisible by 4 without remainder if the last two digits are divisible by 4.
   - An integer is divisible by 5 without remainder if the last digit is divisible by 5.
   - An integer is divisible by 6 without remainder if its checksum is divisible by 3 and the last digit by 2.
   - An integer is divisible by 8 without remainder if the last three digits are divisible by 8.
   - An integer is divisible by 9 without remainder if its checksum is divisible by 9.
   - An integer is divisible by 10 without remainder if the last digit is a 0.
   - For divisibility by 7, there is unfortunately no such simple condition.

1	2	3	4	5	6	7	8	9	10	digits at
1	2	1	2	5	2	1	2	1	0	cond. 5,10,2,4,6,8
3	4	3	4		4	3	4	3		
7	6	7	6		6	7	6	7		
9	8	9	8		8	9	8	9		
1	4	1	2	5	4	1	2	1	0	cond. digits 4 and 8
3	8	3	6		8	3	6	3		
7		7				7		7		
9		9				9		9		
1	4	1	2	5	8	1	2	1	0	digits 1-3 and 4-6
3	8	3	6	5	4	3	6	3		
7		7				7		7		
9		9				9		9		
1	4	1	2	5	8	1	6	1	0	digits 8-10
3	8	3	2	5	8	9	6	3		
7		7	6	5	4	3	2	7		
9		9	6	5	4	7	2	9		
1	4	1	2	5	8	9	6	3	0	digits 4-6 and 7-9
3	8	3	6	5	4	3	2	1		
7		7	6	5	4	3	2	7		
9		9	6	5	4	7	2	3		
			6	5	4	7	2	9		
1	4	7	2	5	8	9	6	3	0	digits 1-3
1	8	3	6	5	4	3	2	1		
1	8	9	6	5	4	3	2	7		
3	8	1	6	5	4	7	2	3		
3	8	7	6	5	4	7	2	9		
7	4	1								
7	8	3								
7	8	9								
9	8	1								
9	8	7								

By consecutive elimination of possibilities, and in the last step the doubles, we get ten possible numbers

1472589630 , 1836547290 , 1896547230 , 1896547290 , 3816547290 , 7412589630 , 7896543210 , 9816543270 , 9816547230 , 9876543210

where only 3816547290 is divisible by 7.

2. Is there always a position on a continuous floor for a rectangular table with four equal legs, such that the table does not wiggle?

   Reason: Fun with the mean value theorem.

   Solution:

   - Quadratic case.

     We consider the heights h of its legs at a certain point x on the floor measured by its angle to a fixed point (radial coordinates). The table doesn't wiggle, if the sum of two opposite heights are equal, i.e. if $h(x)+h(x+\pi)=h(x+\frac{\pi}{2})+h(x+\frac{3\pi}{2})$. The function $f\,:\,\mathbb{R}\longrightarrow\mathbb{R}$ defined by

     $$f(x)=h(x)-h(x+\frac{\pi}{2})+h(x+\pi)-h(x+\frac{3\pi}{2})$$

     is continuous and we assume that our table wiggles, so w.l.o.g. $f(x_0)>0$. Thus

     $$\begin{aligned}0>-f(x_0)&=-h(x_0)+h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})\\&=-h(x_0+2\pi)+h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})\\&=h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})-h(x_0+2\pi)\\&=f(x_0+\frac{\pi}{2})\end{aligned}$$

     With $f(x_0+\frac{\pi}{2})<0<f(x_0)$ we get a point $\xi\in[x_0,x_0+\frac{\pi}{2}]$ such that $f(\xi)=0$ by the mean value theorem, and the table does not wiggle there.

   - Rectangular case.

     Now we have to consider $f(x):=h(x)+h(x+\pi)-h(x+d)-h(x+d+\pi)$ for an angle $d\in(0,\pi)$. By the periodicity of $h(x)$ we get for $H:=\int_0^{2\pi}h(x)h(x)\,dx$ that $H=\int_0^{2\pi}h(x+c)\,dx$ for all

$c \in \mathbb{R}$. Hence

$$\begin{aligned}\int_0^{2\pi} f(x)\,dx &= \int_0^{2\pi} h(x)\,dx + \int_0^{2\pi} h(x+\pi)\,dx \\ &\quad - \int_0^{2\pi} h(x+d)\,dx - \int_0^{2\pi} h(x+d+\pi)\,dx \\ &= H + H - H - H \\ &= 0\end{aligned}$$

and by the mean value theorem for integration there is a point $\xi \in [0, 2\pi]$ with

$$0 = \int_0^{2\pi} f(x)\,dx = f(\xi)(2\pi - 0) \Longrightarrow f(\xi) = 0$$

3. Two mathematicians meet by chance on the plane: "Did not you have three sons?" asks one, "how old are they?" "The product of years is 36," is the answer, "and the sum of years is exactly today's date." "Hmm, that's not enough for me," says the colleague. "Oh, right," says the second mathematician, "I forgot to mention that my eldest son has a dog." How old are the three sons?

   Reason: Logic Puzzle.

   Solution: There are eight combinations for the product, as each son is at least 1. One of them adds to 38=1+1+36 which is no date. Since product and sum weren't sufficient, there have to be two combinations with the same sum, which leaves two possibilities: 1+6+6=2+2+9, from which only one has an oldest son. He is 9.

4. Find functions $y(t)$ and $z(t)$ which locally solve the equations

$$\begin{cases} e^t + \tan y(t) &= 1 \\ t^2 + z(t)^3 + z(t) &= 0 \end{cases}$$

   in a neighborhood of $t = 0$ and investigate their behavior with respect to monotony (where defined). It is sufficient to determine the functions up to a differential equation. It's a mathematical problem, so existence will do.

   Reason: Implicit Function Theorem.

   Solution: The equation system is solvable at $t = 0$ with $y(0) = z(0) =$

0. Furthermore is $\vec{g}(t,\vec{x}) = \begin{pmatrix} e^t + \tan y(t) \\ t^2 + z(t)^3 + z(t) \end{pmatrix}$ with $\vec{x} = (y,z)^\tau$ continuously differentiable with

$$\frac{\partial \vec{g}}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial z} \\ \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+y^2} & 0 \\ 0 & 3z^2+1 \end{bmatrix}$$

with determinant one at $(t,\vec{x}) = (0,0,0)$. By the implicit function theorem, we thus have a neighborhood $U = U_\varepsilon(0) = (-\varepsilon,\varepsilon)$ and functions $y(t), z(t) \in C^1(U)$ which solves our equations for all $t \in (-\varepsilon,\varepsilon)$. Additionally we get for $\vec{f} = (y(t),z(t))^\tau$

$$\begin{aligned} \frac{\partial \vec{f}}{\partial t} &= (y(t)', z(t)')^\tau \\ &= -\left(\frac{\partial \vec{g}}{\partial \vec{x}}(t,\vec{f}(t))\right)^{-1} \cdot \frac{\partial \vec{g}}{\partial t}(t,\vec{f}(t)) \\ &= -\begin{bmatrix} 1+y(t)^2 & 0 \\ 0 & 3z(t)^2+1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^t \\ 2t \end{bmatrix} \\ &= -\begin{bmatrix} \dfrac{1}{1+y(t)^2} & 0 \\ 0 & \dfrac{1}{3z(t)^2+1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^t \\ 2t \end{bmatrix} \\ &= -\left[\frac{e^t}{1+y(t)^2}\,,\ \frac{2t}{3z(t)^2+1}\right]^\tau \end{aligned}$$

Since $y(t)' < 0$ on $(-\varepsilon,\varepsilon)$ the function $y(t)$ is strictly monotone decreasing, whereas the function $z(t)$ due to the nominator of $z(t)'$ is strictly monotone increasing on $(-\varepsilon,0)$ and strictly monotone decreasing on $(0,\varepsilon)$.

5. Areas and Volumes.

   (a) Show that the paraboloid

   $$P = \{\,(x,y,z)^\tau \in \mathbb{R}^3 \,|\, x^2+y^2 = z\,,\ x,y \in [-1,1]\,\} \subseteq \mathbb{R}^3$$

   and the hyperboloid

   $$H = \{\,(x,y,z)^\tau \in \mathbb{R}^3 \,|\, x^2-y^2 = z\,,\ x,y \in [-1,1]\,\} \subseteq \mathbb{R}^3$$

   have equal areas.

(b) Bring $M = \{ (x,y,z)^\tau \in \mathbb{R}^3 \,|\, x^2 \leq y^4 \leq z^8 \leq 1 \}$ into a normal form and calculate its volume.

Reason: Integration.

Solution:

(a) We choose the parameterization $\Phi(u,v) = (u,v,u^2+v^2)^\tau$, $u,v \in [-1,1]$ for P and get the normal vector

$$\begin{aligned}\vec{N}_P(u,v) &= \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\\ &= (1,0,2u)^\tau \times (0,1,2v)^\tau\\ &= (-2u,-2v,1)^\tau\end{aligned}$$

and we get for the area

$$|P| = \int_\Phi 1\, d\sigma = \int_{[-1,1]^2} ||\vec{N}_P(u,v)||\, d(u,v) = \int_{[-1,1]^2} \sqrt{4u^2+4v^2+1}\, d(u,v)$$

The parameterization of H is given by $\Psi(u,v) = (u,v,u^2-v^2)^\tau$ whose normal vector $\vec{N}_H(u,v) = (-2u,2v,1)^\tau$ has the same norm as $\vec{N}_P$ and thus yields the same area integral.

(b) $x \mapsto \sqrt[n]{x}$ is strictly monotone increasing for all even n on $[0,\infty)$ so

$$\begin{aligned} z^8 \leq 1 &\Longleftrightarrow z \in [-1,1]\\ y^4 \leq z^8 &\Longleftrightarrow y \in [-z^2,z^2]\\ x^2 \leq y^4 &\Longleftrightarrow x \in [-y^2,y^2]\end{aligned}$$

A normal form of M is thus given by

$$M = \{ (x,y,z)^\tau \in \mathbb{R}^3 \,|\, -1 \leq z \leq 1 \ , \ -z^2 \leq y \leq z^2 \ , \ -y^2 \leq x \leq y^2 \}$$

and

$$\begin{aligned}
|M| &= \int_M d(x,y,z) \\
&= \int_{-1}^{1}\int_{-z^2}^{z^2}\int_{-y^2}^{y^2} dx\,dy\,dz \\
&= \int_{-1}^{1}\int_{-z^2}^{z^2} [x]_{-y^2}^{y^2}\, dy\,dz \\
&= \int_{-1}^{1}\int_{-z^2}^{z^2} 2y^2\,dy\,dz \\
&= \int_{-1}^{1}\left[\frac{2}{3}y^3\right]_{-z^2}^{z^2} dz \\
&= \int_{-1}^{1}\frac{4}{3}z^6\,dz \\
&= \left[\frac{4}{21}z^7\right]_{-1}^{1} \\
&= \frac{8}{21}
\end{aligned}$$

6. The table cards at a rotatable round table with 12 seats are set up for expected 12 people. However, the persons ignore the cards and randomly distribute themselves to the seats.

   Is it always possible with a single turn of the table to make sure that at least two people sit in front of their table cards?

   Reason: Modular arithmetics.

   Solution: If we denote the distance of a guest to his expected seat by $\pi(i)-i$ for the given permutation $\pi \in S_{12}$, then the question is: Are there always two numbers i,j such that for any $\pi \in S_{12}$ we have $\pi(i)-i=\pi(j)-j\,?$

   Assume this is not the case. Then $\{\,\pi(i)-i\,|\,0<i<13\,\}\cong \mathbb{Z}_{12}$ and $\sum_{i=1}^{12}(\pi(i)-i)=0$. On the other hand is $\sum_{\mathbb{Z}_{12}} i = 78 \equiv 6 \bmod 12$ which cannot both be true.

7. Kummer and Bertrand.

   Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be a sequence of positive real numbers and $A = \sum_{n=1}^{\infty} a_n$. Prove the following statements:

(a) If there is a sequence $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ of positive real numbers, such that there is an index N for which $b_{n-1} \cdot \dfrac{a_{n-1}}{a_n} - b_n \geq C$ for a constant $C > 0$ and all $n > N$, then A converges.

(b) If there is a sequence $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ of positive real numbers, such that the series $\sum_{n=1}^{\infty} \dfrac{1}{b_n}$ diverges, and there is an index $N \in \mathbb{N}$ such that $b_{n-1} \cdot \dfrac{a_{n-1}}{a_n} - b_n \leq 0$ for all $n > N$, then A diverges.

(c) We define the sequence of real numbers by

$$b_n := \left(n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) \log(n)$$

and $B := \lim_{n\to\infty} b_n \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\, \pm\infty \,\}$.

Then A converges if $B > 1$ and diverges if $B < 1$.

Solution:

(a) (Kummer's convergence criterion)

Given $0 < C \cdot a_n \leq b_{n-1}a_{n-1} - b_n a_n$ for all $n > N$ we get

$$0 < C \cdot \sum_{n=N+1}^{M} a_n \leq \sum_{n=N+1}^{M} (b_{n-1}a_{n-1} - b_n a_n) = b_N a_N - b_M a_M < b_N a_N$$

and thus $\sum_{n=N+1}^{M} a_n < \dfrac{b_N a_N}{C}$ and we have a sequence of partial sums $A_M = \sum_{n=1}^{M} a_n$ which is strictly monotone increasing for $M > N$ and bounded from above.

(b) (Kummer's divergence criterion)

Our condition now reads $0 < b_N a_N \leq \ldots \leq b_{n-1}a_{n-1} \leq b_n a_n$ for all $n > N$, resp. $a_n \geq \dfrac{b_N}{b_n} a_N$. Hence $\sum_{n=N+1}^{M} a_n \geq b_N a_N \sum_{n=N+1}^{M} \dfrac{1}{b_n}$ which diverges for $M \to \infty$ and so does A by the minority criterion.

(c) (Bertrand's criterion)

Let $c_n := n\log(n)$ for $n > 1$. The series $\sum_{n=2}^{\infty} \dfrac{1}{c_n}$ diverges by the integral criterion ($\int \dfrac{1}{x\log(x)}\,dx \sim \log\log x$). For $f(x) := \dfrac{1}{x\log(x)}$ we have $f(n) = c_n$ and $f(x)$ is monotone decreasing for $x \geq 2$ with $\lim_{x\to\infty} f(x) = 0$. Furthermore we have

$$\lim_{R\to\infty}\int_2^R \frac{1}{x\log(x)}\,dx = \lim_{R\to\infty}\int_2^R \frac{\frac{d}{dx}\log(x)}{\log(x)}\,dx$$
$$= \lim_{R\to\infty}\left[\log(\log(R)) - \log(\log(2))\right] = \infty$$

Now we define

$$\begin{aligned}
K_n &:= c_n \cdot \frac{a_n}{a_{n+1}} - c_{n+1} \\
&= n\log(n)\cdot\frac{a_n}{a_{n+1}} - (n+1)\log(n+1) \\
&= n\log(n)\cdot\frac{a_n}{a_{n+1}} - n\log(n+1) - \log(n+1) \\
&= n\log(n)\cdot\frac{a_n}{a_{n+1}} - n\left(\log\left(1+\frac{1}{n}\right)+\log(n)\right) - \left(\log\left(1+\frac{1}{n}\right)+\log(n)\right) \\
&= n\log(n)\cdot\frac{a_n}{a_{n+1}} - (n+1)\log\left(1+\frac{1}{n}\right) - n\log(n) - \log(n) \\
&= \log(n)\cdot\left(n\frac{a_n}{a_{n+1}} - n - 1\right) - \log\left(1+\frac{1}{n}\right)^{n+1} \\
&= B_n - \log\left(1+\frac{1}{n}\right)^{n+1}
\end{aligned}$$

Since the logarithm is continuous, we get

$$K := \lim_{n\to\infty} K_n = B - \log(e) = B - 1 \in \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$$

and we can apply Kummer's criteria (see previous parts) and A converges if $K > 0$, that is $B > 1$, and diverges, if $K < 0$, that is $B < 1$.

8. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in \mathbb{R}[x]$ be a polynomial where all roots are negative. Prove that

$$\int_1^{\infty} \frac{1}{p(x)}\,dx$$

converges absolutely if and only if $n > 1$.

Reason: Integrals.

Solution: We start with $n \geq 2$.

$$\lim_{x\to\infty} \frac{p(x)}{x^n} = \lim_{x\to\infty} \left(1 + a_{n-1}\frac{1}{x} + a_{n-2}\frac{1}{x^2} + \ldots + a_0\frac{1}{x^n} \right) = 1$$

Multiplication by $\sqrt{x}$ yields

$$\lim_{x\to\infty} \frac{p(x)}{x^{n-\frac{1}{2}}} = \lim_{x\to\infty} \sqrt{x} \left(1 + a_{n-1}\frac{1}{x} + a_{n-2}\frac{1}{x^2} + \ldots + a_0\frac{1}{x^n} \right) = \infty$$

so there is a point $x_0 \in [0, \infty)$ such that $p(x) > x^{n-\frac{1}{2}}$ $(*)$ for all $x > x_0$, especially $p(x) > 0$ in this range. For the integral of absolute values we get

$$\int_1^\infty \left| \frac{1}{p(x)} \right| dx = \int_1^{x_0} \frac{dx}{|p(x)|} + \int_{x_0}^\infty \frac{dx}{|p(x)|} = C + \int_{x_0}^\infty \frac{dx}{|p(x)|} \qquad (**)$$

The integral $\int_{x_0}^\infty x^{\frac{1}{2}-n}\, dx$ converges because $\frac{1}{2} - n < -1$. From $(*)$ we get that this integral is a convergent majorant for the second term in $(**)$ which therefore converges absolutely.

For $\deg(p) = n = 0$ we have $p(x) = 1$ and

$$\int_1^\infty \frac{dx}{p(x)} = \int_1^\infty dx = \lim_{\xi\to\infty} [x]_1^\xi = \infty$$

For $\deg(p) = n = 1$ we have $p(x) = x + c$ and

$$\int_1^\infty \frac{dx}{p(x)} = \int_1^\infty \frac{dx}{x+c} = \lim_{\xi\to\infty} [\log|x+c|]_1^\xi = \infty$$

We have $c > 0$ and the logarithm is defined, for otherwise $p(-c) = 0$ and $p(x)$ would have a positive root which is against our assumption.

9. Let $\emptyset \neq U \subseteq \mathbb{R}^+$ be an open set, and $x_0 \in U$. We define the **quotient logarithm** of a function $f : U \longrightarrow \mathbb{R}^+$ at $x = x_0$ by

$$f^{\,-}(x_0) := \lim_{x\to x_0} \frac{\log f(x) - \log f(x_0)}{\log x - \log x_0}$$

Solve the *differential equation* $f^- = f$.

Reason: Funny differential.

Solution: By the mean value theorem for the logarithm, we find $\xi \in (x, x_0)$, $\eta \in (f(x), f(x_0)$ such that

$$\begin{aligned} f^-(x_0) &= \lim_{x\to x_0} \frac{\log f(x) - \log f(x_0)}{\log x - \log x_0} \\ &= \lim_{x\to x_0} \frac{\frac{1}{\eta}\,(f(x) - f(x_0))}{\frac{1}{\xi}\,(x - x_0)} \\ &= \frac{x_0}{f(x_0)} \cdot f'(x_0) \end{aligned}$$

So we get

$$\begin{aligned} f &= f^- \\ f &= \frac{x}{f}\, f' \\ f^2 &= x\, f' \\ \frac{df}{dx} &= \frac{f^2}{x} \\ \int \frac{df}{f^2} &= \int \frac{dx}{x} \\ -\frac{1}{f} &= \log x + \log c\ ,\ c \in \mathbb{R}^+ \\ f &= -\frac{1}{\log cx} \end{aligned}$$

10. Let $p > q$ be prime numbers such that $p \not\equiv 1 \bmod q$.
Prove that each group with pq elements is cyclic.

Reason: Sylow's theorems.

Solution: Let $n \geq 1$ be the number of Sylow p-subgroups of G, and $m \geq 1$ be the number of Sylow q-subgroups of G.

(i) By Sylow's third theorem $n\,|\,q$ and $n \equiv 1 \bmod p$. Since 1 and q are the only divisors of q, and $1 < q < p$ we can rule out $n = q$ and conclude $n = 1$.

(ii) Similarly we have $m\,|\,p$ and $m \equiv 1 \bmod q$. The condition $p \not\equiv 1 \bmod q$ rules out $m = p$ so we can conclude $m = 1$.

(iii) Say H is the Sylow p-subgroup and K the Sylow q-subgroup.
Since gHg^{-1} is a Sylow p-subgroup, and gKg^{-1} is a Sylow q-subgroup,

too, both subgroups H, K have to be normal, and $HK \leq G$. Because $H, K \subsetneq HK$, i.e. $p, q \mid |HK| \mid |G| = pq$ with $|HK| > p+1 > q+1$ we get $|HK| = pq$, resp. $G = HK$. We also have $|H \cap K| = 1$ as $|H \cap K| \,|\, |H| = p$ and $|H \cap K| \,|\, |K| = q$, so $G = H \times K$ is a direct product of normal subgroups H, K of prime order p, resp. q. However, groups of prime order are cyclic and we get

$$G = H \times K = \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$$

11. If we multiply our five digits number by four, we will get the same number in reverse order. What's the number?

    Reason: Arithmetic Riddle.

    Solution: Let x be the number we are looking for. Since $x, 4x$ both have five digits, x has to be between $9,999$ and $25,000$, i.e. x will start with a one or a two. As $4x$ has to be even, we get $x = 2----$ and $4x \geq 80,000$. So the last digit of x has to be eight or nine. As $4x$ ends with a two, we have $x = 2---8$. The second digit of x cannot create an overflow or we would have $4x \geq 90,000$. Thus the second digit of x is in $\{0,1,2\}$ so we are looking for an $x \in \{20--8, 21--8, 22--8\}$ with $4x \in \{8--02, 8--12, 8--22\}$. Say $x = ---c8$, then $4x = y + (40c + 30 + 2)$ and testing for $4c + 3 \in \{0,1,2\}$ yields $c \in \{2,7\}$ and with $4 \cdot 28 = 112$, $4 \cdot 78 = 312$ a second but last digit one, i.e. $x = 21--8$. Furthermore the second but last digit of x has to be two or seven. If it was a two, then $4x = 82-12$ but $x > 21,000$ and $4x > 84,000$. Thus we have $x = 21-78$ and $4x = 87-12$. From $4x = 87c12 = 4 \cdot 21c78 = 84,000 + 400c + 312$ we get $30 + c = 4c + 3$ or $c = 9$ and $x = 21,978$.

12. Let $\mathcal{B}$ be a Boolean ring with 1, i.e. each element of $\mathcal{B}$ is idempotent. Show that each prime ideal is maximal.

    Reason: Abstract Algebra.

    Solution: Let $P \subseteq \mathcal{B}$ be a prime ideal. Then $\mathcal{B}/P$ is an integral domain. We show that $\mathcal{B}/P \cong \mathbb{Z}_2$ which is a field, and therefore P is maximal.

    Let $x, y \in \mathcal{B}/P - \{0\}$ and $z := x \cdot y$. Then $xz = x(xy) = (xx)y = xy$ and $0 = x(z-y)$. As $\mathcal{B}/P$ is an integral domain and $x \neq 0$, we have $y = z$ and similarly $x = z$. So all elements different from 0 are identical. Because $P \neq \mathcal{B}$ we get $\mathcal{B}/P \cong \mathbb{Z}_2$.

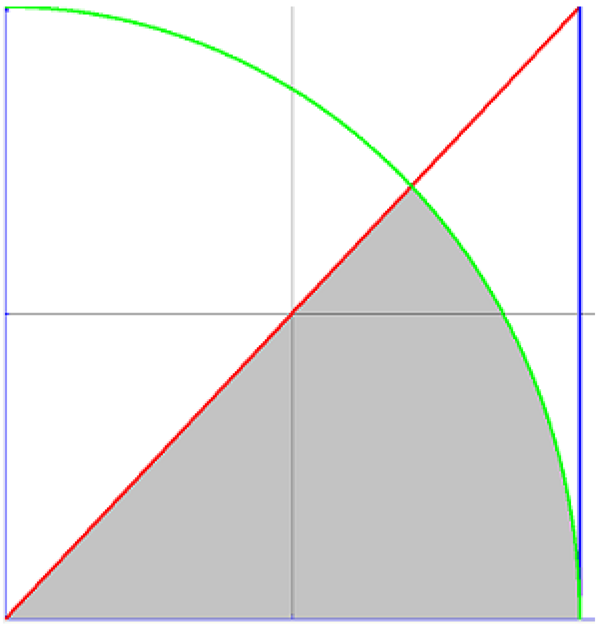
13. On a state fair is a booth where you can shoot at a square. You get 1 point for each hit and 2 points if you hit closer to the center than to the boundary. How big is your chance to get the extra point?

    Reason: Geometry.

    Solution: We want to derive the curve within the square which marks the limiting condition. Think of a Cartesian coordinate with the square's center as its origin, and a side length of 2 . We get for a point $P=(x,y)$ which lies in the first quadrant and fulfills the boundary condition

$$1-x=\sqrt{x^2+y^2}\Longrightarrow y=\pm\sqrt{1-2x}$$

On the diagonal $x=y$ it is the point $y=x=\sqrt{2}-1=:x_0$ and $(0.5\,,\,1)$ on the $x-$axis.



The area A is therefore

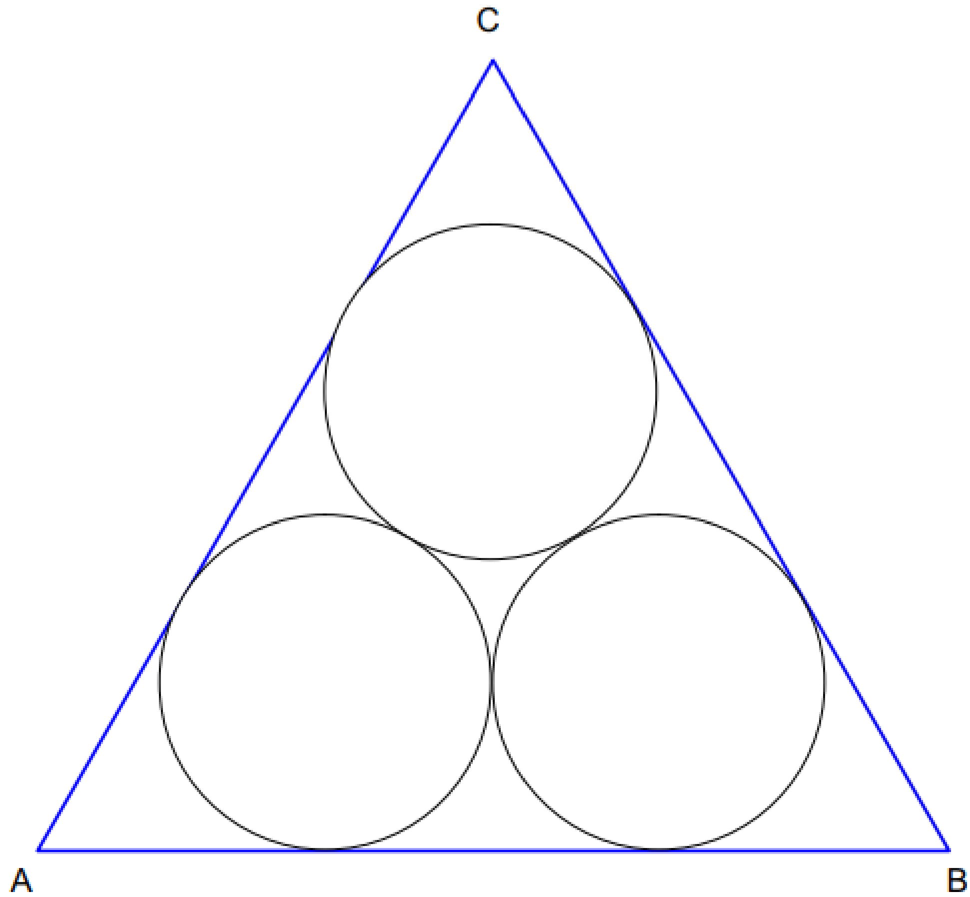
$$\begin{aligned}A&=\int_0^{x_0}x\,dx+\int_{x_0}^{\frac{1}{2}}\sqrt{1-2x}\,dx\\&=\frac{1}{2}x_0^2+\frac{1}{3}\left(1-2x_0\right)^{\frac{3}{2}}\\&=\frac{3}{2}-\sqrt{2}+\frac{5}{3}\sqrt{2}-\frac{7}{3}\\&=\frac{2}{3}\sqrt{2}-\frac{5}{6}\end{aligned}$$

As we have 8 such areas and the square has an area of 4, we get a total chance of

$$p = \frac{8}{4}\left(\frac{2}{3}\sqrt{2} - \frac{5}{6}\right) = \frac{4}{3}\sqrt{2} - \frac{5}{3} \approx 0.218951416... \approx 21.9\,\%$$

14. The Italian mathematician G.F.Malfatti presented the following in 1803, because of its degree of difficulty well-known task: Construct three circles into a given triangle so that the total area of the circles is maximal.

    For the equilateral triangle, Malfatti found the solution



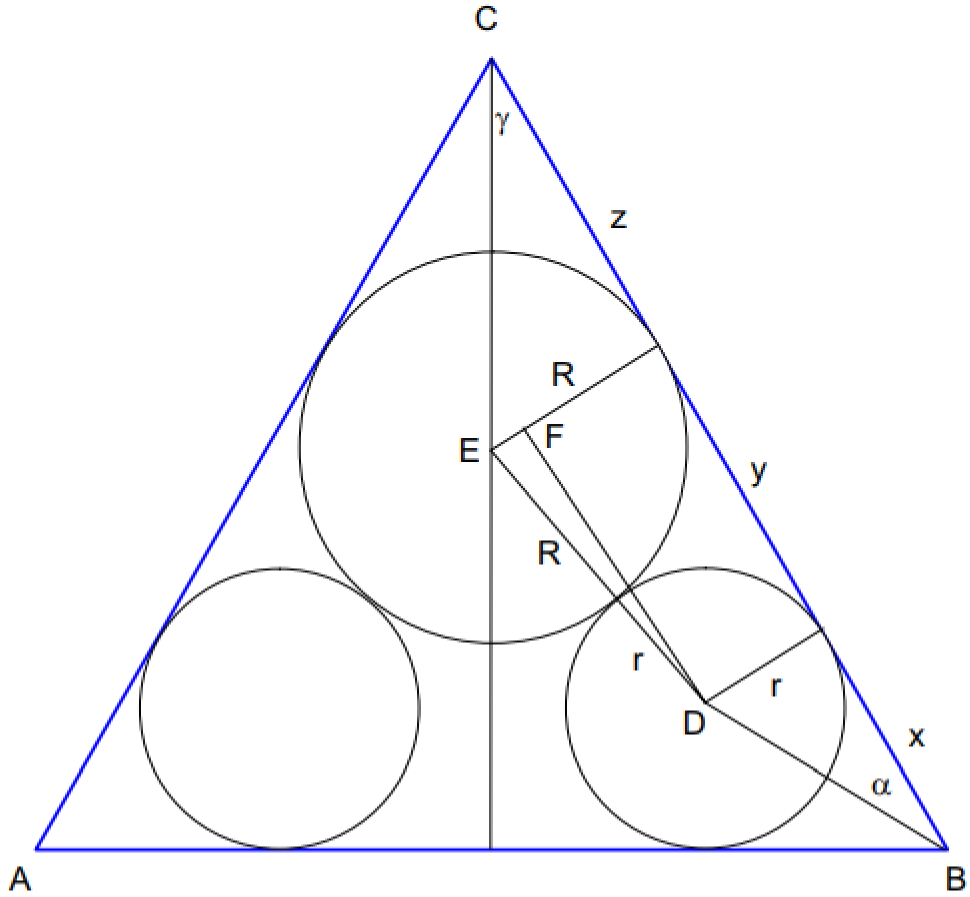
It wasn't until 1929 when the mathematicians Lob and Richmond showed that Malfatti had made a mistake here.

Show that there is a better solution for the equilateral triangle.

Reason: Geometry.

Solution: Problem and graphics from
http://www.matheraetsel.de/archiv/Extremwerte/Malfatti1/malfatti1_2.pdf.

We examine all symmetric solutions in which
a) each circle center lies on an angle bisector and
b) two circles touch each other.



We calculate the distances x and z with the tangent theorem in a right triangle

$$\tan\alpha = \frac{r}{x}\ ,\ \alpha = \frac{1}{6}\pi\ ,\ x = r\sqrt{3} \qquad (1) \tag{1}$$

$$\tan\gamma = \frac{R}{z}\ ,\ \gamma = \frac{1}{6}\pi\ ,\ z = R\sqrt{3} \qquad (2) \tag{2}$$

From the touching condition of the circles we get a right triangle DEF with $y = DF$ such that

$$(R+r)^2 = (R-r)^2 + y^2 \longrightarrow y = 2\sqrt{Rr} \tag{3}$$

The sum of all three sections x, y, z are the side length of the triangle

$$\overline{BC} =: a = x + y + z = r\sqrt{3} + 2\sqrt{Rr} + R\sqrt{3} \tag{4}$$

In case of equal circles, i.e. $r = R$ we then have $R = \dfrac{a}{2(1+\sqrt{3})}$ and for the sum of their areas

$$A_1 = 3 \cdot \pi R^2 = \frac{3\pi}{4(1+\sqrt{3})^2}a^2 \approx 31.567\,\%\,a^2$$

Next we chose R to be the radius of the inscribed circle, which for an equilateral triangle is

$$R_0 = \frac{a}{2\sqrt{3}} \tag{5}$$

and with equation (4)

$$a = r\sqrt{3} + 2\sqrt{\frac{ar}{2\sqrt{3}}} + \frac{a}{2} \Longrightarrow r = \frac{a}{6\sqrt{3}} \tag{6}$$

For the sum of all three discs we thus get

$$A_2 = \pi R_0^2 + 2\pi r^2 = (\frac{\pi}{12} + \frac{\pi}{54})a^2 = \frac{11\pi}{108}a^2 \approx 31.9977\,\%\,a^2 \tag{7}$$

15. Find all three digits numbers $x = [abc] = 100a + 10b + c$ such that all powers $x^n\ ,\ n \in \mathbb{N}$ end on $[...abc]$, too.

    Reason: Puzzle.

    Solution: The problem is trivial for $n = 1$ so let us assume we had solved it for $n = 2$. Then $x^3 = [abc]^3 = [abc] \cdot [abc]^2 = [abc] \cdot [...abc] =$

$[abc] \cdot (1000d + 100a + 10b + c)$ and we are back to the power of two, as the multiplication by $1000d$ doesn't contribute anything to the last three digits. Thus it is sufficient to solve the problem for $n = 2$.

$x^2 = (100a + 10b + c)^2 = 10 \cdot d + c^2$ and the only digits which end up squared by the same digit are $c \in \{0, 1, 5, 6\}$.

(a) $c = 0$.
$x^2 = (100a + 10b)^2 = 10000a^2 + 2000ab + 100b^2$
The last but one digit is zero, so $b = 0$ and $x^2 = 10000a^2$ which means for the second before last digit $a = 0$, but $x = 0$ hasn't three digits.

(b) $c = 1$.
$x^2 = (100a+10b+1)^2 = 10000a^2+2000ab+200a+100b^2+20b+1$
Thus $b = 2b \bmod 10$, i.e. $b = 0$ and $x^2 = 1000d + 200a + 1$.
Therefore $2a = a \bmod 10$ or $a = 0$ which isn't possible.

(c) $c = 5$.
$(100a + 10b + 5)^2 = 10000a^2 + 2000ab + 1000a + 100b^2 + 100b + 25$
The result shows, that $b = 2$ so $x^2 = 1000d + 625$ which is a solution, since $625^2 = 390625$.

(d) $c = 6$.
$(100a + 10b + 6)^2 = 10000a^2 + 2000ab + 1200a + 100b^2 + 120b + 36$
The result shows, that $b \in \{1, 3, 5, 7, 9\}$ is an odd number. Squaring $[b6]$ yields, that only $[76]$ is a possible solution, i.e. $b = 7$ and $x^2 = 1000d + 1200a + 776$. Again we thus have only odd values for a and testing them leaves $[abc] = 376$.

Therefore there are two three digits numbers which fulfill the condition: 625 and 376 .

16. The teacher writes a number less than 50,000 on the board.

The first student finds that n is divisible by 2.
The second student finds that n is divisible by 3.
The third student finds that n is divisible by 4.
...
The twelfth student finds that n is divisible by 13.

Ten of the students told the truth, two lied. The two liars have made their statements immediately after each other.

What number did the teacher write on the board?

Reason: Logic Puzzle.

Solution: The least common multiple of $1, 2, 3, \ldots, 13$ is $360,360$. The following students couldn't have lied, because it resulted in another lie of a student who is not their neighbor:

$$\begin{array}{rlrlrl} n=2 & (4) & n=3 & (6) & n=4 & (8) \\ n=5 & (10) & n=6 & (12) & n=10 & (2 \text{ or } 5) \\ n=12 & (3 \text{ or } 4) & & & & \end{array}$$

Thus the only combination of liars left are $(7,8)$ and $(8,9)$.
$(360,360 : 7) : 2 = 25,740$ and $(360,360 : 3) : 2 = 60,060 > 50,000$, i.e. the teacher wrote $25,740$ on the board.

17. Calculate curvature and torsion of the curve

$$x : [0,a] \longrightarrow \mathbb{R}^3\, ,\, x(t) = \left(t, t^2, \frac{2}{3}t^3\right)^\tau$$

Reason: Physicist's Practice.

Solution: Since $\dfrac{d\sigma}{dt} = ||\dot{x}(t)||_2 = \sqrt{1+4t^2+4t^4} = 1+2t^2$ we get for the tangent unit vector

$$T(t) = \frac{dx}{d\sigma} = \frac{dx/dt}{d\sigma/dt} = \frac{1}{1+2t^2}\begin{pmatrix} 1 \\ 2t \\ 2t^2 \end{pmatrix}$$

$$\frac{dT}{d\sigma} = \frac{dT/dt}{d\sigma/dt} = \frac{1}{(1+2t^2)^3} = \begin{pmatrix} -4t \\ 2-4t^2 \\ 4t \end{pmatrix}$$

The curvature is therefore

$$\kappa(t) = \left|\left|\frac{dT}{d\sigma}\right|\right|_2 = \frac{(16t^2 + 4(1-2t^2)^2 + 16t^2)^{\frac{1}{2}}}{(1+2t^2)^3} = \frac{2}{(1+2t^2)^2}$$

the normal vector

$$N(t) = \frac{1}{\kappa(t)} \cdot \frac{dT}{d\sigma} = \frac{1}{1+2t^2}\begin{pmatrix} -2t \\ 1-2t^2 \\ 2t \end{pmatrix}$$

and the binormal vector

$$B(t) = T(t) \times N(t) = \frac{1}{1+2t^2}\begin{pmatrix} 2t^2 \\ -2t \\ 1 \end{pmatrix}$$

From that we get

$$\frac{dB}{d\sigma} = \frac{dB/dt}{d\sigma/dt} = \frac{1}{(1+2t^2)^3}\begin{pmatrix} 4t \\ 4t^2-2 \\ -4t \end{pmatrix}$$

and the torsion is

$$\tau(t) = -\frac{dB}{d\sigma}\cdot N(t) = \frac{2}{(1+2t^2)^2}$$

18. In which country in Europe originated this custom? Write down the numbers behind your answers from left to right and post your result.

    Reason: Quiz.

    Solution: $27, 81, 81, 9, 24, 34, 12, 14, 22, 23\ :\ \text{for } n = 0, 1, ..., 5$
    $2^n + 7^n + 8^n + 18^n + 19^n + 24^n = 3^n + 4^n + 12^n + 14^n + 22^n + 23^n$

    (a) From Nikulden to Budni Vecher is lent in this country. At Christmas you can then taste Kravai, the traditional Christmas bread. The presents on Christmas Eve brings the Djado Koleda ("Grandfather Christmas").

    Russia 56 - Bulgaria 27 : Bulgaria 27

    (b) Christmas is Jul and a house elf named Nisse (Julenisse) is even more important in this country than Santa Claus. It is said that he lives in stables and in barns and takes care of the animals there. He likes to play a little prank on the children.

    Norway 87 - Denmark 81 : Denmark 81

    (c) The Christmas holidays are also called "beer festivals" in this country. Traditionally, they were celebrated rather quietly in the circle of the family. Even visitors were rather undesirable on the Christmas holidays, female visit on the 2nd Christmas holiday was once even considered a particularly bad omen. Christmas dinner in this country includes dishes such as roast goose, sauerkraut, potatoes, or ginger cookies.

    Estonia 81 - Germany 42 : Estonia 81

    (d) Ilex and mistletoe are important symbols of Christmas in this country, as is the robin that is most often seen on Christmas cards.

    England 9 - Netherlands 3 : England 9

(e) At Christmas, a log is burned in the fireplace and a cake shaped like a log is made, according to old customs. Otherwise, you will dine in this country rather nobly with selected delicacies. Even the smell of roasted sweet chestnuts must not be missing in the run-up to Christmas.

France 24 - Spain 67 : France 24

(f) An absolute must at Christmas in this country is the Joulukinkku, the Christmas ham. Christmas peace is proclaimed in this country on 12/24 and deceased family members are remembered on Christmas Eve. For Christmas dinner you will be served traditional rice pudding with cinnamon, sugar and an almond, which should bring good luck.

Latvia 16 - Finland 34 : Finland 34

(g) This country put the Christmas tree into the focus of Christmas for the first time. Also edible tree decoration of former times was replaced for the first time by glass balls. There are not just one, but several official gift bringers.

Germany 12 - Czech Republic 2 : Germany 12

(h) 14 days before Christmas, they turn up, the thirteen charming, but also somewhat sneaky Christmas goblins - the last one, called the "Thirteenth", will not disappear until January 6th. However, the children of this country have to be especially careful of the troll woman GrÃ½la, the mother of the thirteen kobolds - who was not good, is caught by her and consumed without further ado. An absolute must at Christmas is the traditional Christmas drink JolaÃ¶l.

Iceland 14 - Greenland 9 : Iceland 14

(i) Christmas is referred to in the language of this country as "Winter Festival" and Christmas Eve is the "Winter Festive Evening". On this day, according to Christian custom, the birth of Christ is celebrated, and according to ancient pagan custom, the return of the virgin of the sun. A popular Christmas ornament is Puzuri, a kind of mobile made of straw.

Latvia 22 - Poland 33 : Latvia 22

(j) An important pre-Christmas custom in this country is the celebration of Lucia Day: on the day of Saint Lucia, December 13, the eldest daughter of the house plays the "Lucia Bride" and wakes the whole family for breakfast. On Christmas Eve, which is often called the "Dopparedan" (one-day's day), sausages, potato casserole with anchovies or Lutfisk, marinated cod are often served. The gifts are not brought by Santa Claus or the Christkind, but the Julbock.

Sweden 23 - Lithuania 31 : Sweden 23

19. Ten DYK Christmas gifts.

Reason: Facts.

Solution:

10: The image of Santa Claus flying his sleigh began in 1819 and was created by Washington Irving,
(https://en.wikipedia.org/wiki/Washington_Irving),
the same author who dreamt up the Headless Horseman.

09: Clement Moore's poem introduced eight more reindeer for Santa's sleigh and their names were Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Duner and Blixem (for the German words for thunder (Donner) and lightning (Blitz)). These later evolved into Donner and Blitzen.

08: Some leave food out for Santa Claus' reindeer as Norse children did, leaving hay and treats for Odin's eight-legged horse Sleipnir hoping they would stop by during their hunting adventures. Dutch children adopted this same tradition, leaving food in their wooden shoes for St. Nicholas' horse.

07: America's first batch of eggnog was made in the Jamestown settlement in 1607. Its name comes from the word "grog", meaning any drink made with rum. Non-alcoholic eggnog is popular as well.

06: Between the 16th and 19th centuries global temperatures were significantly lower than normal in what was known as a "little ice age". Charles Dickens grew up during this period and experienced snow for his first eight Christmases. This "White Christmas" experience influenced his writing and began a tradition of expectation for the holidays.

05: The Christmas wreath
(https://en.wikipedia.org/wiki/Wreath#Advent_and_Christmas_wreaths)
was originally hung as a symbol of Jesus. The holly represents his crown of thorns and the red berries the blood he shed.

04: Tinsel was invented in 1610 in Germany and was once made of real silver.

03: A Christmas tree
(https://en.wikipedia.org/wiki/Christmas_tree)
is a decorated tree, usually an evergreen conifer such as spruce, pine, or fir or an artificial tree of similar appearance, associated with the celebration of Christmas. The modern Christmas tree was developed in medieval Livonia (present-day Estonia and Latvia) and early modern Germany, where Protestant Germans brought decorated trees into their homes. It acquired popularity beyond the Lutheran areas of Germany and the Baltic countries during the second half of the 19th century, at first among the upper classes.

02: The tradition of hanging stockings comes from a Dutch legend. A poor man had three daughters for whom he could not afford to provide a dowry. St. Nicholas dropped a bag of gold down his chimney and gold coins fell out and into the stockings drying by the fireplace. The daughters now had dowries and could be married, avoiding a life on the streets.

01: In 1914 during World War I there was a now famous Christmas truce
(https://en.wikipedia.org/wiki/Christmas_truce)
in the trenches between the British and the Germans. They exchanged gifts across a neutral no man's land, played football together, and decorated their shelters.

2 October-B 2018

1. Calculate

(a)
$$\sum_{n=0}^{\infty}\left(\frac{2}{2+3i}\right)^n$$

(b)
$$\sum_{n=0}^{\infty}\left(2\sqrt{n}-4\sqrt{n+1}+2\sqrt{n+2}\right)$$

(c)
$$\sum_{n=3}^{\infty}\frac{8n}{(n^2-1)^2}$$

(d)
$$\lim_{x\to 0}\frac{\cos(x^2)-\sqrt{1+x^3}}{x^3}$$

Reason: Easy.

Solution:

(a)

$$\begin{aligned} S_n &= \sum_{k=0}^{n}\left(\frac{2}{2+3i}\right)^k \\ &= \left[1-\left(\frac{2}{2+3i}\right)^{n+1}\right]\cdot\left[1-\frac{2}{2+3i}\right]^{-1} \\ &= \left[1-\left(\frac{2}{2+3i}\right)^{n+1}\right]\cdot\frac{2+3i}{3i} \end{aligned}$$

Because we have $\left|\frac{2}{2+3i}\right| = \frac{2}{\sqrt{13}} < 1$ we get

$$\sum_{n=0}^{\infty}\left(\frac{2}{2+3i}\right)^n = \frac{1}{1-\frac{2}{2+3i}} = \frac{2+3i}{3i} = 1-\frac{2}{3}i$$

(b) It follows by induction that the $n-$th partial sum is

$$S_n = -2(1 + \sqrt{n+1} - \sqrt{n+2})$$

so we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(2\sqrt{n} - 4\sqrt{n+1} + 2\sqrt{n+2}\right) &= \lim_{n\to\infty} S_n \\ &= -2 - 2\cdot \lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n+2}) \\ &= -2 + 2\cdot \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n+2})} \\ &= -2\end{aligned}$$

(c) With the partial fraction $\dfrac{8n}{(n^2-1)^2} = \dfrac{2}{(n-1)^2} - \dfrac{2}{(n+1)^2}$ we have

$$\begin{aligned}S_n &= \sum_{k=3}^{n} \frac{4k}{(k^2-1)^2} \\ &= 2\sum_{k=3}^{n}\left(\frac{1}{(k-1)^2} - \frac{1}{(k+1)^2}\right) \\ &= 2\left(\sum_{k=2}^{n-1}\frac{1}{k^2} - \sum_{k=4}^{n+1}\frac{1}{k^2}\right) \\ &= \frac{13}{18} - \frac{2}{n^2} - \frac{2}{(n+1)^2}\end{aligned}$$

and $\sum_{n=3}^{\infty} \dfrac{8n}{(n^2-1)^2} = \lim_{n\to\infty} S_n = \dfrac{13}{18}$

(d) We have the Taylor expansions $\cos(x^2) = 1 - \dfrac{x^4}{2} + O(x^6)$ and

$\sqrt{1+x^3} = 1 + \frac{x^3}{2} - \frac{x^6}{8} + O(x^9)$ at $x = 0$ and thus

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x^2) - \sqrt{1+x^3}}{x^3} &= \lim_{x\to 0} \frac{1 - \frac{x^4}{2} + O(x^6) - 1 - \frac{x^3}{2} + \frac{x^6}{8} + O(x^9)}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{x^3}{2} - \frac{x^4}{2} + \frac{x^6}{8} + O(x^6) + O(x^9)}{x^3} \\
&= \lim_{x\to 0} \left(-\frac{1}{2} - \frac{1}{2}x + \frac{1}{8}x^3 + O(x^3) + O(x^6) \right) \\
&= -\frac{1}{2} + \lim_{x\to 0} O(x^3) \\
&= -\frac{1}{2}
\end{aligned}$$

2. (a) Determine $\int_1^\infty \frac{\log(x)}{x^3}\,dx$.
   (b) Determine for which α the integral $\int_0^\infty x^2 \exp(-\alpha x)\,dx$ converges.
   (c) Find a sequence of functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$ such that

$$\sum_{\mathbb{N}} \int_{\mathbb{R}} f_n(x)\,dx \neq \int_{\mathbb{R}} \left(\sum_{\mathbb{N}} f_n(x) \right) dx$$

   (d) Find a family of functions $f_r : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $r \in \mathbb{R}$ such that

$$\lim_{r\to 0} \int_{\mathbb{R}} f_r(x)\,dx \neq \int_{\mathbb{R}} \lim_{r\to 0} f_r(x)\,dx$$

   (e) Find an example for which

$$\frac{d}{dx} \int_{\mathbb{R}} f(x,y)\,dy \neq \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy$$

Reason: Integrals and commutativity with other operations.

Solution:

(a) By partial integration ($u = \log(x)$, $v' = x^{-3}$) we get

$$\begin{aligned}\int_1^\infty \frac{\log(x)}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\log(x)}{x^3}\,dx \\ &= -\lim_{t\to\infty}\frac{\log(x)}{2x^2}\Big|_1^t + \lim_{t\to\infty}\int_1^t \frac{1}{2x^3}\,dx \\ &= -\lim_{t\to\infty}\frac{\log(t)}{2t^2} - \lim_{t\to\infty}\frac{1}{4x^2}\Big|_1^t \\ &= \frac{1}{4}\end{aligned}$$

(b) For $\alpha = 0$ we get an infinite integral. Let us now assume $\alpha \neq 0$. By partial integration twice, we get

$$\begin{aligned}\int_0^\infty x^2\exp(-\alpha x)\,dx &= -\frac{x^2\exp(-\alpha x)}{\alpha}\Big|_0^\infty + \frac{2}{\alpha}\int_0^\infty x\exp(-\alpha x)\,dx \\ &= -\frac{x^2\exp(-\alpha x)}{\alpha}\Big|_0^\infty \\ &+ \frac{2}{\alpha}\left(-\frac{x}{\alpha}\exp(-\alpha x)\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty \exp(-\alpha x)\,dx\right) \\ &= \frac{\exp(-\alpha x)}{\alpha}\left[-x^2 - \frac{2}{\alpha}x - \frac{2}{\alpha^2}\right]_0^\infty \\ &= \frac{2}{\alpha^3} - \lim_{x\to\infty}\exp(-\alpha x)\left(\frac{x^2}{\alpha} + \frac{2x}{\alpha^2} + \frac{2}{\alpha^3}\right) \\ &= \begin{cases}\dfrac{2}{\alpha^3} & \text{if } \alpha > 0 \\ \text{not existent} & \text{if } \alpha \leq 0\end{cases}\end{aligned}$$

(c) With $f_n = \operatorname{Ind}[n, n+1] - \operatorname{Ind}[n+1, n+2]$ we have $\int_{\mathbb{R}} f_n(x)\,dx = 0$ and $\sum_{\mathbb{N}} f_n(x) = \operatorname{Ind}[1,2]$ and so

$$0 = \sum_{\mathbb{N}}\int_{\mathbb{R}} f_n(x)\,dx \neq \int_{\mathbb{R}}\sum_{\mathbb{N}} f_n(x)\,dx = 1$$

(d) With $f_r(x) = \begin{cases}0 & \text{if } |x| \geq r \\ \frac{r-|x|}{r^2} & \text{if } |x| < r\end{cases}$ we have $\int_{\mathbb{R}} f_r(x)\,dx = 1$ for $r > 0$ and $\lim_{r\to 0} f_r(x) = f_0(x) = \begin{cases}\infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0\end{cases}$ and so

$$1 = \lim_{r\to 0}\int_{\mathbb{R}} f_r(x)\,dx \neq \int_{\mathbb{R}}\lim_{r\to 0} f_r(x)\,dx = 0$$

(e) We first define $g(t) = \exp(-t^2)$ and show $I := \int_{\mathbb{R}} g(x)\,dx = \sqrt{\pi}$.

$$\begin{aligned} I^2 &= \left(\int_{\mathbb{R}} g(x)\,dx\right) \cdot \left(\int_{\mathbb{R}} g(y)\,dy\right) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)g(y)\,dx\,dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(-x^2-y^2) \\ &= \int_0^\infty \int_0^{2\pi} r\exp(-r^2)\,d\varphi\,dr \\ &= 2\pi \int_0^\infty r\exp(-r^2)dr \\ &= -\pi\left[\exp(-r^2)\right]_0^\infty \\ &= \pi \end{aligned}$$

The function $h(x,y) = x \cdot g(xy)$ is continuous, however the parameter integral $H(x) = \int_{\mathbb{R}} h(x,y)\,dy$ is not:

$$\begin{aligned} H(x) &= \int_{\mathbb{R}} x \cdot g(xy)\,dy \\ &= \operatorname{sgn}(x) \int_{\mathbb{R}} |x| \cdot g(xy)\,dy \\ &= \operatorname{sgn}(x) \int_{\mathbb{R}} g(t)\,dt \\ &= \operatorname{sgn}(x) \cdot \sqrt{\pi} \end{aligned}$$

Now let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ the continuously differentiable function $f(x,y) = x|x|g(xy)$ such that we for $F(x) = \int_{\mathbb{R}} f(x,y)\,dy$

$$\begin{aligned} F(x) &= \int_{\mathbb{R}} x|x|g(xy)\,dy \\ &= x\int_{\mathbb{R}} |x|g(xy)\,dy \\ &= x\int_{\mathbb{R}} g(t)dt \\ &= x\sqrt{\pi} \qquad \text{and} \\ F'(x) &= \sqrt{\pi} \end{aligned}$$

Differentiation under the integral gives us

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy = \int_{\mathbb{R}} \left(2|x|g(xy) + x|x|\frac{d}{dx}g(xy)\right)\,dy$$

which vanishes at $x = 0$. For $x \neq 0$ we get

$$\begin{aligned}\int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy &= 2\int_{\mathbb{R}} g(t)\,dt - 2\int_{\mathbb{R}} t^2 g(t)\,dt \\ &= 2\sqrt{\pi} - 2\left(\left[-\frac{t}{2}\exp(-t^2)\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{\mathbb{R}} g(t)\,dt\right) \\ &= \sqrt{\pi}\end{aligned}$$

In the end we have

$$\sqrt{\pi} = \left.\frac{d}{dx}\right|_{x=0} F(x) = \left.\frac{d}{dx}\right|_{x=0} \int_{\mathbb{R}} f(x,y)\,dy \neq \int_{\mathbb{R}} \left.\frac{\partial}{\partial x}\right|_{x=0} f(x,y)\,dy = 0$$

and although both sides are identical for $x \neq 0$ they differ at $x = 0$ and thus cannot be the same.

3. (a) Show that $D_4 = \langle r, s \,|\, r^2 = s^2 = rsrs = 1 \rangle$ is the smallest non-cyclic group.
   (b) Show that the converse of Lagrange's theorem is false, i.e. that there is a finite group with n elements which has no subgroup to one of the divisors of n .
   (c) Give an example of a non-Abelian finite and a non-Abelian infinite group.
   (d) Show that A_5 is simple, i.e. has only trivial normal subgroups.

Reason: Groups.

Solution:

(a) $D_4 = V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$
D_4 has four elements, but none is of order four, so it cannot by cyclic. Smaller groups have necessarily one ($\{1\}$), two ($\mathbb{Z}_2$) or three ($\mathbb{Z}_3$) elements and are all cylic.

(b) A_4 has no subgroup of order 6.
$6 \,|\, |A_4| = 12$. Since A_4 consists of eight $3-$cycles, three pairs of disjoint transpositions, and 1, a subgroup of order six has to contain at least one $3-$cycle, say (123). Now as a subgroup of index two, it would also be normal. By conjugation of (123) with all pairs of transpositions, and inversion, we would get all other $3-$cycles as elements of this subgroup. But a subgroup of at least eight elements cannot have six.

(c) $S_3 \cong D_6$ and $GL_n(\mathbb{R})$
$D_6 = \langle r, s \,|\, s^2 = r^3 = srsr = 1 \rangle$ and reflexion s and rotation r do not commute.
The center of $GL_n(\mathbb{R})$ is $Z(GL_n(\mathbb{R})) = \operatorname{Diag}_n(\mathbb{R}) \subsetneq GL_n(\mathbb{R})$.

(d) We use a similar argument as in part (b), namely:

If a normal subgroup $N \trianglelefteq A_n$, $n > 2$, *contains a* $3-$*cycle, then* $N = A_n$.

Proof: Say we have $(123) \in N$ and so are $(123)^{-1} = (132) \in N$ and $\sigma(132)\sigma^{-1} \in N$ for $\sigma \in A_n$. Thus

$$(12)(3k)(132)(3k)^{-1}m(12)^{-1} = (12k) \in N \ , \ k > 3$$

However, A_n is generated by all $3-$cycles of the form $(12k)$. (Ex.)

Now let us assume we have a normal subgroup $\{1\} \triangleleft N \triangleleft A_5$. For our proof which works for any group A_n , $n > 4$ we choose a permutation $1 \neq \tau \in N$ which leaves the maximal number of elements invariant, resp. permutes a minimal number of elements, and show, that τ has to be a $3-$cycle. The statement then follows by the above.

Assume τ moves four elements, which means it has to be w.l.o.g. $\tau = (12)(34)$. Then

$$N \ni \tau[(345)\tau(345)^{-1}] = (12)(34)(345)(12)(34)(354) = (345)$$

which permutes only three elements in contradiction to the minimality of τ.
Thus let us assume τ permutes more than four numbers. Then we can write w.l.o.g. $\tau = (12345)$ as only possibility to permute all five numbers, since all other possibilities involve odd permutations. (The general case $n > 5$ has to consider more possibilities. The trick then is to write τ by starting with the longest cycle on the left.) Again we get

$$N \ni (234)\tau(234)^{-1} = (234)(12345)(243) = (13425) \neq \tau$$

and $N \ni \tau^{-1}(13425) = (43215)(13425) = (124)$ leaving $3, 5$ invariant in contradiction to the minimality of τ .

4. One tiny nocturnal and long-living beetle decided one night to climb a sequoia. The tree was exactly $100\,m$ high at this time. Every night the beetle made a distance of $10\,cm$. The tree grew every day evenly

$20\,cm$ along its entire length.
Did the beetle eventually reach the top of the tree? And if so, how many nights will he need at least?

Reason: Riddle.

Solution: On the first night the beetle manages 10/10000 of the tree trunk. On the second night he crawls 10/10020 of the tree trunk. On the third night he crawls 10/10040 of the tree trunk. He has reached the top when the sum of the track parts reaches 1.

$$\sum_{n=0}^{N} \frac{1}{1000+2n} \geq 1$$
$$\sum_{n=0}^{N} \frac{1}{500+n} \geq 2$$
$$\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{499} \frac{1}{n} \geq 2$$
$$H_N - H_{499} \geq 2$$

A close estimation is $H_n = \gamma + \log(n) + \varepsilon$ with the Euler-Mascheroni constant γ and a small error ε. A numerical solution yields at least $3691 - 499 = 3192$ nights.

3 October-I 2018

1. (a) Let X be a set and $\mathcal{F} = \{\,\{\,x\,\}\,|\,x \in X\,\}$. Determine the $\sigma-$algebra $\sigma(\mathcal{F})$.
   (b) Let X be a set and $S \subseteq \mathcal{P}(X)$. Show that for $A \in \sigma(S)$ there is an $S_0 \subseteq S$ such that S_0 is countable and $A \in \sigma(S_0)$.

Reason: Analysis.

Solution: A $\sigma-$algebra $\sigma(\mathcal{B})$ over a set $\mathcal{B}$ is a subset of the power set $P(\mathcal{B})$, which contains $\mathcal{B}$ as an element or equivalently $\emptyset$, is closed under complements to $\mathcal{B}$, and closed under the union of countable many sets. For an arbitrary set X and a family of subsets $B \subseteq \mathcal{P}(X)$, $\sigma(B)$ denotes the intersection of all $\sigma-$algebras of subsets of X that contain B, i.e. $\sigma(B) = \cap\{\,\sigma(C)\,|\,C \subseteq B\,\}$ with complements are taken to X.

(a) We define $\mathcal{A} := \{ A \subseteq X \,|\, A \text{ is countable or } X - A \text{ is countable } \}$ and show $\mathcal{A} = \sigma(\mathcal{F})$.

Every countable set $A = \{ x_i \,|\, i \in \mathbb{N} \} = \bigcup_{i\in\mathbb{N}} \{ x_i \} \in \sigma(\mathcal{F})$ and also each set with a countable complement. Thus $\mathcal{A} \subseteq \sigma(\mathcal{F})$. In order to show $\sigma(\mathcal{F}) \subseteq \mathcal{A}$, we show that $\mathcal{A}$ is a $\sigma-$algebra which contains $\mathcal{F}$.

i. $\emptyset \in \mathcal{A}$ since it is countable.

ii. If $A \in \mathcal{A}$ then $X - A \in \mathcal{A}$ since $A = X - (X - A)$.

iii. Let $A_i \in \mathcal{A}$, $i \in \mathbb{N}$. Then either all A_i are countable, and then $\cup_{i\in\mathbb{N}} A_i$ is countable, too, and in $\mathcal{A}$, or there is an index j with an uncountable set A_j. In this case is $X - A_j$ countable. Then we have that $X - \cup_{i\in\mathbb{N}} A_i = \cap_{i\in\mathbb{N}} X - A_i \subseteq X - A_j$ is countable, and again $\cup_{i\in\mathbb{N}} A_i \in \mathcal{A}$.

iv. If $A = \{ x \} \in \mathcal{F}$, then it is countable and so is $A \in \mathcal{A}$, i.e. $\mathcal{F} \subseteq \mathcal{A}$.

(b) Let $\mathcal{A} = \bigcup \{ \sigma(C) \,|\, C \subseteq S \text{ countable} \}$. We show that $\mathcal{A}$ is a $\sigma-$algebra over X which contains S.

i. $\emptyset \in \sigma(\emptyset) = \{\emptyset, X\}$ and $\emptyset$ is countable with $\emptyset \subseteq S$, so $\emptyset \in \mathcal{A}$.

ii. With $A \in \mathcal{A}$ we have a countable set $C \subseteq S$ with $A \in \sigma(C)$, and so is $X - A \in \mathcal{A}$.

iii. Let $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$. Then there are countable sets $C_i \subseteq S$ with $A_i \in \sigma(C_i)$. The set $C = \bigcup_{i\in\mathbb{N}} C_i \subseteq S$ is also countable. Since $C_i \subseteq C$ and σ is monotone, we have $A_i \in \sigma(C)$ and so $\bigcup_{i\in\mathbb{N}} A_i \in \sigma(C) \in \mathcal{A}$.

If $A \in S$, then $\{ A \} \subseteq S$ is countable and $A \in \sigma(\{A\}) = \{ \emptyset, A, X - A, X \}$, i.e. $A \in \mathcal{A}$ and thus $S \subseteq \mathcal{A}$.

Since $\mathcal{A}$ is a $\sigma-$algebra which contains S, we have $\sigma(S) \subseteq \mathcal{A}$. This means that for all $A \in \sigma(S)$ there is a countable set $S_0 \subseteq S$ with $A \in \sigma(S_0)$.

2. (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \leq y < x+1 \\ -1 & \text{if } x \geq 0 \text{ and } x+1 \leq y < x+2 \\ 0 & \text{elsewhere} \end{cases}$$

Now calculate $\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, d\lambda(x) \right] \, d\lambda(y)$ and $\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, d\lambda(y) \right] \, d\lambda(x)$, and why isn't it a contradiction to Fubini's theorem.

(b) Show that the integral

$$\int_A \frac{1}{x^2+y}\,d\lambda(x,y)$$

with $A=(0,1)\times(0,1)\subseteq\mathbb{R}^2$ is finite.

Reason: Fubini.

Solution:

(a) For a certain $x>0$ we get

$$\int_{\mathbb{R}} f(x,y)\,d\lambda(y) = \int_x^{x+1} 1\,d\lambda(y) - \int_{x+1}^{x+2} 1\,d\lambda(y) = 1-1=0$$

and especially $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,d\lambda(y)\,d\lambda(x)=0$

Now we fix a certain $y\in\mathbb{R}$. The condition $x\leq y<x+1$ means $y-1<x\leq y$ and $x+1\leq y<x+2$ means $y-2<x\leq y-1$. However, we also have the condition $x\geq 0$ so we have to distinguish $y\in[0,1)$, $y\in[1,2)$, $y\in[2,\infty)$. We therefore get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,d\lambda(x)\,d\lambda(y) = \\
&= \int_{[0,1)}\int_{\mathbb{R}} f(x,y)\,d\lambda(x)\,d\lambda(y) + \int_{[1,2)}\int_{\mathbb{R}} f(x,y)\,d\lambda(x)\,d\lambda(y) + \\
&+ \int_{[2,\infty)}\int_{\mathbb{R}} f(x,y)\,d\lambda(x)\,d\lambda(y) \\
&= \int_{[0,1)}\int_0^y 1\,dx\,dy + \int_{[1,2)} 1 - \int_0^{y-1} 1\,dx\,dy + \int_{[2,\infty)} 0\,dy \\
&= \int_{[0,1)} y\,dy + \int_{[1,2)} (1-(y-1))\,dy \\
&= \frac{1}{2}+2-\frac{3}{2} \\
&= 1
\end{aligned}$$

Fubini's theorem does not apply, because $f(x,y)$ isn't continuous, resp. not non-negative a.e.

(b) The function $f(x,y)=\dfrac{1}{x^2+y}$ is positive on $A=(0,1)\times(0,1)$ so we may apply the theorem of Fubini and get

$$\int_0^1\int_0^1 \frac{1}{x^2+y}\,dy\,dx = \int_0^1 \log(x^2+y)\Big|_{y=0}^{y=1}\,dx = \int_0^1 [\log(x^2+1)-\log(x^2)]\,dx$$

The logarithm function is monotone increasing and positive on $[1,2]$ so $\int_0^1 \log(x^2+1)\,dx \leq \log 2$. Furthermore we have

$$\int_0^1 \log(x^2)\,dx = 2\cdot [x\log(x)-x]_0^1 = -2$$

and $\int_A \dfrac{1}{x^2+2}\,d\lambda(x,y) \leq 2+\log 2 < \infty$.

3. Let $\hat{\mathbb{C}} = \mathbb{C}\cup\{\,\infty\,\}$ with the usual Euclidean topology on $\mathbb{C}$ and

$$\hat{\mathcal{T}} = \{\,U\subseteq\hat{\mathbb{C}}\,|\,\infty\notin U \wedge U\subseteq\mathbb{C} \text{ open}\,\}\cup\{\,U\subseteq\hat{\mathbb{C}}\,|\,\infty\in U\wedge U^C\subseteq\mathbb{C}\text{ compact}\,\}$$

(a) $\hat{\mathcal{T}}$ is a topology on $\hat{\mathbb{C}}$.
(b) $(\hat{\mathbb{C}},\hat{\mathcal{T}})$ is Hausdorff.
(c) $(\hat{\mathbb{C}},\hat{\mathcal{T}})$ is compact.

Reason: Alexandroff Extension.

Solution:

(a) Since $\emptyset$ is open and compact, we have $\emptyset,\hat{\mathbb{C}}\in\hat{\mathcal{T}}$.

Now let $(O_\alpha)_{\alpha\in I}$ be a family of sets in $\hat{\mathcal{T}}$ and $O := \cup_{\alpha\in I}O_\alpha$.

If $\infty\notin O_\alpha$ for all $\alpha\in I$ then by definition the O_α are open in $\mathbb{C}$ and so is O with $\infty\notin O$. This means that $O\in\hat{\mathcal{T}}$.

Now assume $\infty\in O_\beta$ for a $\beta\in I$. Then

$$O^C = O^C\cap\mathbb{C} = \bigcap_{\alpha\in I}(O_\alpha^C\cap\mathbb{C})\subseteq O_\beta^C$$

and each of the sets $O_\alpha^C\cap\mathbb{C}$ is closed in $\mathbb{C}$ because of $O_\alpha\in\hat{\mathcal{T}}$. But then O^C is a closed subset of a compact set in $\mathbb{C}$ and thus also compact. Since $\infty\in O$ we have $O\in\hat{\mathcal{T}}$.

Now let $(O_i)_{i=1}^n$ be finitely many sets in $\hat{\mathcal{T}}$ and $O := \cap_{i=1}^n O_i$.

Then $O_i\cap\mathbb{C}\subseteq\mathbb{C}$ is open for every $1\leq i\leq n$. Is there a j with $\infty\notin O_j$ then $\infty\notin O$ and

$$O = O\cap\mathbb{C} = \cap_{i=1}^n(O_i\cap\mathbb{C})$$

is open in $\mathbb{C}$ and thus $O\in\hat{\mathcal{T}}$.

If $\infty\in O_i$ for all i, then $\infty\in O$ and $O^C = \cup_{i=1}^n O_i^C\subseteq\mathbb{C}$ is a finite union of compact sets compact in $\mathbb{C}$ again, i.e. $O\in\hat{\mathcal{T}}$.

(b) Let $x \neq y \in \hat{\mathbb{C}}$. If $x, y \in \mathbb{C}$ then we have disjoint open neighborhoods $U_x, U_y \subseteq \mathbb{C}$ by the Hausdorff property of $\mathbb{C}$, and they are also disjoint sets in $\hat{\mathcal{T}}$. Now let $y = \infty$, $x \in \mathbb{C}$. We set

$U_x = \{ z \in \mathbb{C} \, : \, |z-x| < 1 \}$ and $U_y = \{ z \in \mathbb{C} \, : \, |z-y| > 1 \} \cup \{ \infty \}$

and get two disjoint sets which contain x, y resp. As both are also in $\hat{\mathcal{T}}$, we have shown that $(\hat{\mathbb{C}}, \hat{\mathcal{T}})$ is Hausdorff.

(c) Let $(O_\alpha)_{\alpha \in I}$ be a family of open sets in $\hat{\mathcal{T}}$ and $\hat{\mathbb{C}} = \cup_{\alpha \in I} O_\alpha$. Let further be $\infty \in O_\beta$. Then $K = O_\beta^C \subseteq \mathbb{C}$ is compact and

$$K = K \cap \mathbb{C} \subseteq \hat{\mathbb{C}} \cap \mathbb{C} = \cup_{\alpha \in I} (O_\alpha \cap \mathbb{C})$$

an open covering of the compact set K. Therefore we have a finite covering $K \subseteq \cup_{i=1}^n O_{\alpha_i}$. This means

$$\hat{\mathbb{C}} = O_\beta \cup K \subseteq O_\beta \cup \bigcup_{i=1}^n O_{\alpha_i}$$

is a finite subcover and $\hat{\mathbb{C}}$ compact.

4. Calculate

$$\int_{-\infty}^{+\infty} \frac{4}{x^2 - x + 1} \, dx$$

Reason: Residue Theorem.

Solution: For a meromorph function $f(x) = \dfrac{P(x)}{Q(x)}$ with polynomials $P, Q \in \mathbb{C}[x]$ with $Q^{-1}(0) \cap \mathbb{R} = \emptyset$ and $\deg Q > 1 + \deg P$ we have as a Corollary of the residue theorem (Ex.)

$$\int_{-\infty}^{+\infty} f(t) \, dt = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}_z(f)$$

which we can apply here as the poles are $z_{1,2} = \frac{1}{2} \pm \frac{1}{2} i \sqrt{3}$ and for $\operatorname{Im}(z_1) > 0$

$$Res_{z_1}(f) = \frac{4}{z_1 - z_2} = \frac{4}{2i \operatorname{Im}(z_1)} = -i \frac{4}{\sqrt{3}}$$

and we get for the integral $\int_{-\infty}^{+\infty} f(t) \, dt = (2\pi i) \cdot (-i \dfrac{4}{\sqrt{3}}) = \dfrac{8\pi}{\sqrt{3}}$

4 August-B 2018

1. Given a nonnegative, monotone decreasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Prove that $\sum_{n\in\mathbb{N}} a_n$ converges if and only if $\sum_{n\in\mathbb{N}_0} 2^n a_{2^n}$ converges.

   Reason: Cauchy's Condensation criterion. (M)

   Solution: Let's assume $\sum_{n\in\mathbb{N}} a_n$ converges. We set $S_n = \sum_{k=1}^{n} a_k$ and calculate

$$\begin{aligned} S_{2^n} &\geq a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{n-1}a_{2^n} \\ &\geq \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n}\right) \\ &= \frac{1}{2}\sum_{k=0}^{2^n} 2^k a_k \end{aligned}$$

   Since $\sum_{k=1}^{\infty} a_k$ converges, so does the series $(S_n)_{n\in\mathbb{N}}$ of partial sums and thus twice the subsequence $2\cdot(S_{2^n})_{n\in\mathbb{N}}$. But this is the boundary from above for the non-negative sums $\sum_{k=1}^{n} 2^k a_k$, i.e. $\sum_{k=1}^{\infty} 2^k a_k$ converges.

   Let now $n < 2^{m+1} - 1$. Then

$$\begin{aligned} \sum_{k=1}^{n} a_k &\leq \sum_{k=1}^{2^{m+1}-1} a_k \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \ldots) + (a_{2^m} + \ldots) \\ &= \sum_{k=0}^{m} 2^k a_k \end{aligned}$$

   If $\sum_{k=0}^{\infty} 2^k a_k$ converges, then $\sum_{k=0}^{\infty} a_k$ is bounded and converges, too.

2. Calculate
$$\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1} \frac{\sqrt{5}^m}{m!}\cdot\left(\frac{(2k)!}{k!}\right)^2 \frac{2^{-6k-2}}{(2k-m+1)!}$$

   Reason: Easy if the series of arccos is given. (T)

Solution:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1} \frac{\sqrt{5}^m}{m!}\cdot\left(\frac{(2k)!}{k!}\right)^2 \frac{2^{-6k-2}}{(2k-m+1)!}\\
=&\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1}\binom{2k}{k}\binom{2k+1}{m}\frac{\sqrt{5}^m}{2k+1}\cdot\frac{1}{4^k\cdot 4^{2k+1}}\\
=&\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{1+\sqrt{5}}{4}\right)^{2k+1}\frac{1}{4^k\cdot(2k+1)}\\
=&\frac{\pi}{2}-\arccos\left(\frac{1+\sqrt{5}}{4}\right)\\
=&\frac{3}{10}\,\pi
\end{aligned}$$

3. Show that the product $P = xyz$ of a Pythagorean triple $x^2 + y^2 = z^2$ is always divisible by $60 \,|\, P$. Since this is an easy problem, please make sure you won't forget to name an argument!

   Reason: Puzzle. (M)

   Solution: We have to show that $3, 4, 5 \,|\, P$.

   (a) $3 \,|\, P$: Squares only can have remainders $0, 1$ by division by three, hence z^2. In case z^2 isn't divisible by three, the remainders of the sum must be 0 and 1 and thus one of x^2, y^2 is divisible by three and since 3 is prime, this is also true for x or y .

   (b) $4 \,|\, P$: All Pythagorean triples are of the form $x = u^2 - v^2$, $y = 2uv$, $z = u^2 + v^2$ so it remains to show, that $2 \,|\, uv(u + v)(u - v)(u^2 + v^2)$, but if both, u and v are odd, then $(u - v)$ and $(u + v)$ are even.

   (c) $5 \,|\, P$: Squares only can have remainders $0, 1, 4$ by division by five, hence z^2. In case z^2 isn't divisible by five, the remainders of the sum must be 0 and 1 and thus one of x^2, y^2 is divisible by five and since 5 is prime, this is also true for x or y .

4. Given two vector fields $v, w : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$v(x,y) = \begin{bmatrix} y \\ x-y \end{bmatrix} \;,\; w(x,y) = \begin{bmatrix} y-x \\ -y \end{bmatrix}$$

and two curves in $\mathbb{R}^2$ given as:
γ_1 is the half circle from $(0,-1)$ to $(0,1)$ with radius one and origin $(0,0)$, run anti-clockwise from bottom to top.
γ_2 is the straight line segments from $(0,-1)$ to $(1,0)$ and from $(1,0)$ to $(0,1)$, also run through from bottom to top.

(a) Compute all 4 path integrals of v and w with both paths γ_1, γ_2 .

(b) Determine whether v or w have potentials.

Reason: For engineers. (E)

Solution: In the first step we parameterize the two curves:

$$\begin{aligned} c_1 &: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \gamma_1 \text{ with } c_1(t) = (\cos t, \sin t) \text{ for } \gamma_1 \text{ and} \\ c_{2,1} &: [0,1] \longrightarrow \gamma_2 \text{ with } c_{2,1}(t) = (t, t-1) \text{ and} \\ c_{2,2} &: [0,1] \longrightarrow \gamma_2 \text{ with } c_{2,2}(t) = (1-t, t) \text{ for } \gamma_2 \end{aligned}$$

and get $\dot{c}_1(t) = (-\sin t, \cos t)$, $\dot{c}_{2,1}(t) = (1,1)$, $\dot{c}_{2,2}(t) = (-1,1)$. Thus

$$\begin{aligned} \int_{\gamma_1} v\,ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(c_1(t)) \cdot \dot{c}_1(t)\,dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t + \cos^2 t - \sin t \cos t\,dt \\ &= \left[-\frac{1}{2}\cos^2 t + 2 \sin t \cos t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_{\gamma_2} v\,ds &= \int_0^1 v(c_{2,1}(t)) \cdot \dot{c}_{2,1}(t)\,dt + \int_0^1 v(c_{2,2}(t)) \cdot \dot{c}_{2,2}(t)\,dt \\ &= \int_0^1 \begin{bmatrix} t-1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} t \\ 1-2t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} dt \\ &= \int_0^1 1 - 2t\,dt \\ &= \left[t - t^2\right]_0^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned}\int_{\gamma_1} w\,ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(c_1(t))\cdot \dot{c}_1(t)\,dt\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix}\sin t-\cos t\\ -\sin t\end{bmatrix}\cdot\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}\,dt\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t\,dt\\ &= \left[-\frac{1}{2}(t-\sin t\cos t)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= -\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{\gamma_2} w\,ds &= \int_0^1 w(c_{2,1}(t))\cdot \dot{c}_{2,1}(t)\,dt + \int_0^1 w(c_{2,2}(t))\cdot \dot{c}_{2,2}(t)\,dt\\ &= \int_0^1 \begin{bmatrix}-1\\ 1-t\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix} + \begin{bmatrix}2t-1\\ -t\end{bmatrix}\cdot\begin{bmatrix}-1\\ 1\end{bmatrix}\,dt\\ &= \int_0^1 1-4t\,dt\\ &= \left[t-2t^2\right]_0^1\\ &= -1\end{aligned}$$

So the vector field v is apparently path independent whereas w is not. We check this by the calculation of their curl.

$$\operatorname{rot}\vec{F}(x,y) = \operatorname{curl}\vec{F}(x,y) = \frac{\partial \vec{F}_y}{\partial x} - \frac{\partial \vec{F}_x}{\partial y} = \begin{cases}0 & \text{if } \vec{F}=v\\ -1 & \text{if } \vec{F}=w\end{cases}$$

So v has a potential and thus is path independent, and w has none.

5. Show that $\mathbb{Z}[x]/\langle x^2+2x+4\;,\;5\rangle \cong \mathbb{Z}_5[4+\sqrt{2}]$ are isomorphic rings.

   Reason: Abstract algebra. (M)

   Solution: For convenience let $\xi = 4+\sqrt{2}$.
   Then $\xi \notin \mathbb{Z}_5$ because $2\in\mathbb{Z}_5$ has no root, and $\overline{m}(x) := x^2+2x+4$ is the minimal polynomial of ξ, since $\overline{m}(\xi)=0$. Hence

$$\mathbb{Z}_5[\xi] \cong \mathbb{Z}_5[x]/\langle x^2+2x+4\rangle$$

with the isomorphism $\varphi \;:\; \mathbb{Z}_5[\xi] \to \mathbb{Z}_5[x]/\langle x^2+2x+4\rangle \;,\; \varphi(\xi)=x\,.$ The additivity is easy to verify and for the multiplication we have

$$\begin{aligned}\varphi((a\xi+b)(c\xi+d)) &= \varphi(ac\xi^2+(bc+ad)\xi+bd\\ &= acx^2+(bc+ad)x+bd\\ &= (3ac+bc+ad)x+(ac+bd)\\ &= 3acx+ac+adx+bcx+bd\\ &= acx^2+adx+bcx+bd\\ &= (ax+b)(cx+d)\\ &= \varphi(a\xi+b)\varphi(b\xi+c)\end{aligned}$$

Surjectivity is given by construction, and for an element $\varphi(a\xi+b) = ax+b \in \langle x^2+2x+4\rangle$ in the kernel, we get

$$\begin{aligned}ax+b &= (cx+d)\cdot(x^2+2x+4)\\ &= cx^3+(d+2c)x^2+(4c+2d)x+4d\\ &= cx(3x+1)+(d+2c)(3x+1)+(4c+2d)x+4d\\ &= 3cx^2+(c+3d+c+4c+2d)x+(d+2c+4d)\\ &= 3c(3x+1)+cx+2c\\ &= 0\end{aligned}$$

and hence $a\xi+b=0\,.$

It remains to show that $\mathbb{Z}[x]/\langle x^2+2x+4\;,\;5\rangle \cong \mathbb{Z}_5[x]/\langle x^2+2x+4\rangle\,.$ Therefore we consider the ideals $5\mathbb{Z}[x] \subseteq \langle m(x)\;,\;5\rangle$ in $\mathbb{Z}[x]$ and apply the second isomorphism theorem for rings and get

$$\mathbb{Z}[x]/\langle m(x)\;,\;5\rangle \cong (\mathbb{Z}[x]/5\mathbb{Z}[x])/(\langle m(x)\;,\;5\rangle/5\mathbb{Z}[x]) \cong \mathbb{Z}_5[x]/\langle \overline{m}(x)\rangle$$

where $m(x)=x^2+2x+4\in\mathbb{Z}[x]$

6. Calculate
$$\int_0^\pi \frac{\sin(\varphi)}{3\cos^2(\varphi)+2\cos(\varphi)+3}\,d\varphi$$

Reason: Weierstrass substitutions. (T)

Solution: To solve this integral we make use of the Weierstrass substitutions, resp. tangent half angle substitutions. We set $t := \tan\left(\frac{1}{2}\varphi\right)$ and get

$$\sin(\varphi)=\frac{2t}{1+t^2}\,,\;\cos(\varphi)=\frac{1-t^2}{1+t^2}\,,\;d\varphi=\frac{2}{1+t^2}\,dt$$

and so

$$\frac{\sin(\varphi)\,d\varphi}{3\cos^2(\varphi)+2\cos(\varphi)+3} = \frac{\dfrac{2t}{1+t^2}\cdot\dfrac{2}{1+t^2}}{3\dfrac{(1-t^2)^2}{(1+t^2)^2}+2\dfrac{1-t^2}{1+t^2}+3}\;dt$$
$$= \frac{4t\;dt}{(3-6t^2+3t^4)+2(1-t^4)+(3+6t^2+3t^4)}$$
$$= \frac{4t}{4t^4+8}\;dt$$
$$= \frac{t}{t^4+2}\;dt$$

With the substitution $u=\frac{1}{2}t^2$ we get

$$\int_0^\pi \frac{\sin(\varphi)}{3\cos^2(\varphi)+2\cos(\varphi)+3}\,d\varphi = \int_0^\infty \frac{t}{t^4+2}\;dt$$
$$= \frac{1}{2\sqrt{2}}\int_0^\infty \frac{du}{1+u^2}$$
$$= \frac{1}{2\sqrt{2}}\left[\arctan\frac{1}{\sqrt{2}}\;t^2\right]_0^\infty$$
$$= \lim_{t\to\infty}\frac{1}{2\sqrt{2}}\arctan(t)$$
$$= \frac{\pi}{4\sqrt{2}}$$

7. Integrate $\int_1^5 \frac{dx}{\sqrt{x^2+3x-4}}$

Reason: Euler Substitution $\sqrt{x^2+3x-4}=(x+4)t$ (T)

Solution: In order to solve this integral, we look at the zeros in the denominator which are $x=-4$ and $x=1$. We choose the former and proceed by an Euler substitution: $\sqrt{x^2+3x-4}=\sqrt{(x+4)(x-1)}=(x+4)t$. We then have

$$x=\frac{1+4t^2}{1-t^2}\;,\;\sqrt{x^2+3x-4}=\frac{5t}{1-t^2}\;,\;dx=\frac{10t}{(1-t^2)^2}\,dt$$

and

$$\begin{aligned}
\int_1^5 \frac{dx}{\sqrt{x^2+3x-4}} &= 2\int_{\cdots}^{\cdots} \frac{dt}{1-t^2} \\
&= 2\int_{\cdots}^{\cdots} \left(\frac{1}{2}\cdot\frac{1}{1-t}+\frac{1}{2}\cdot\frac{1}{1+t}\right) dt \\
&= \left[\log\left|\frac{1+t}{1-t}\right|\right]_{\cdots}^{\cdots} \\
&= \left[\log\left|\left(1+\frac{\sqrt{x-1}}{\sqrt{x+4}}\right)\left(1-\frac{\sqrt{x-1}}{\sqrt{x+4}}\right)^{-1}\right|\right]_1^5 \\
&= \left[\log\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|\right]_1^5 \\
&= \log 5 \approx 1.61
\end{aligned}$$

8. The random waiting time X on a telephone hotline is characterized by the distribution function F with

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ a - be^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

for some parameters $a, b, \lambda \in \mathbb{R}$ with $\lambda > 0$. We further assume that $P(X = 0) = 0.5$, i.e. there is a 50 % chance not to wait at all, and $P(X > 1[min]) = 0.25$.

Reason: (E)

(a) Determine the parameters with the given information, such that F is actually a distribution function.

Solution: For a distribution function we need $\lim_{x\to\infty} F(x) = 1$ which means $a = 1$. Since $F(x) = 0$ for $x < 0$ and $P(X = 0) = \frac{1}{2}$ we have $F(0) = a - b = 1 - b = \frac{1}{2}$ and thus $b = \frac{1}{2}$. Finally we have

$$F(1) = 1 - \frac{1}{2}e^{-\lambda} = P(X \leq 1) = 1 - P(X > 1) = \frac{3}{4}$$

hence $\lambda = \log 2$ and $F(X) = 1 - \frac{1}{2}e^{-\log(2)\, X}$.

(b) Can the distribution be described by a density function? Why? If yes, calculate the density function.

Solution: Because the distribution function isn't continuous (at $x = 0$) and thus not differentiable, it cannot result from a density function.

9. A princess decided one day to go swimming in the circular lake far from the castle of her father. As soon as she got into the water, suddenly a witch appeared, who wanted to kidnap the girl. The princess swam quickly into the middle of the lake to think of an escape plan. She noticed three things:

   - The witch can run four times as fast as I can swim.
   - The witch always tries to stay close to me.
   - On land, I'm faster than the witch.

   Is there a way for the princess to escape, how? Why doesn't she have a chance to escape?

   Reason: Riddle. (M)

   Solution: The princess swims a bit towards the witch. Once there, she begins to swim in concentric circles. This allows her to move further and further away from the witch, as her angular velocity is higher than that of the witch. Once she has reached the maximum possible distance to the witch in this constellation (the princess and the witch are on a straight line that goes through the center of the pond), she floats on the shortest possible path to the shore. She reaches the shore in front of the witch.

   So we have to determine what the maximal radius of the girl's circle is (being still faster), and whether this will be sufficient to reach the shore in time. We must have with the radius R of the lake

$$\omega_P > \omega_W \iff \frac{v_P}{R_P} > \frac{v_W}{R} = \frac{4v_P}{R} \iff R_P < \frac{1}{4}R$$

So we have to show that for the remaining time

$$\begin{aligned} T_W &= \frac{\text{half circle}}{\text{speed}} = \frac{\pi R}{v_W} \\ &> \frac{3R}{v_W} = \frac{\frac{3}{4}R}{\frac{1}{4}v_W} \\ &= \frac{R - \frac{1}{4}R}{v_P} = \frac{\text{remaining distance}}{v_P} \\ &= \frac{R - R_P}{v_P} = T_P \end{aligned}$$

10. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x,y) = x(x-1)^2 - 2y^2$. Determine all critical points of f, decide whether there are extrema, and which, and at last consider, whether f has global extrema or not.

    Reason: Extrema. (E)

    Solution: We have $\nabla f(x,y) = ((x-1)(3x-1), -4y)$ and the necessary condition $\nabla f = 0$ yields the points $(1,0)$ and $(\frac{1}{3}, 0)$. The Hesse matrix of f is $\begin{bmatrix} 6x-4 & 0 \\ 0 & -4 \end{bmatrix}$ and with the second partial derivative test we conclude that $(x,y) = (1,0)$ is a saddle point and $(x,y) = (\frac{1}{3}, 0)$ a local maximum. Since $\lim_{x \to \pm\infty} f(x,0) = \pm\infty$ there are no global extrema.

5 August-I 2018

1. Let R be a ring with identity element 1 and $r \in R$ an element without left inverse but at last one right inverse $r \cdot a_0 = 1$. Prove that there are infinitely many right inverses to r.

   Reason: Ring theory. (T)

   Solution: Let $a_0, \ldots, a_N$ be all distinct right inverse elements of r. Set

   $$c_k := a_0 + 1 - a_k \cdot r \ , \ 0 \leq k \leq N.$$

   Then $rc_k = r \cdot a_0 + r \cdot 1 - r \cdot a_k \cdot r = 1 + r - 1 \cdot r = 1$ and all c_k are right inverse to r. Next let us assume $c_i = c_j$. This means $a_i r = a_j r$ and multiplying by a_0 from the right yields $a_i = a_j$ which means $i = j$ since we assumed all a_k to be different. Thus all c_k are distinct, too. Now $a_0 \neq c_k$ for otherwise a_k would be a left inverse to r which doesn't exist. Hence $\{ a_0, c_0, \ldots, c_N \}$ are $N + 2$ distinct right inverse elements of r, so we have found one more than assumed, which means there are infinitely many of them.

2. Consider the Lie algebra of skew-Hermitian 2×2 matrices $\mathfrak{g} := \mathfrak{su}(2, \mathbb{C})$ and the Pauli matrices (note that Pauli matrices are not a basis!)

   $$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Now we define an operation on $V := \mathbb{C}_2[x, y]$, the vector space of all complex polynomials of degree less than three in the variables x, y by

   $$\begin{aligned} \varphi(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3).(a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy) = \\ = x(-i\alpha_1 a_3 + \alpha_2 a_3 - \alpha_3 a_1) + \\ + x^2(2i\alpha_1 a_5 + 2\alpha_2 a_5 + 2\alpha_3 a_2) + \\ + y(-i\alpha_1 a_1 - \alpha_2 a_1 + \alpha_3 a_3) + \\ + y^2(2i\alpha_1 a_5 - 2\alpha_2 a_5 - 2\alpha_3 a_4) + \\ + xy(-i\alpha_1 a_2 - i\alpha_1 a_4 + \alpha_2 a_2 - \alpha_2 a_4) \end{aligned}$$

   Show that

   (a) an adjusted φ defines a representation of $\mathfrak{su}(2, \mathbb{C})$ on $\mathbb{C}_2[x, y]$

   (b) Determine its irreducible components.

(c) Compute a vector of maximal weight for each of the components.

Reason: Representation of $\mathfrak{su}(2,\mathbb{C})$. (M)

Solution: The solution might look a bit long due to the necessary matrix calculations, but I think it's worth having an explicit example of a $\mathfrak{su}(2,\mathbb{C})$ representation.

We start with the second point as we can get strong hints by inspection. The constant polynomials are obviously sent to zero, so $\mathbb{C}\cdot 1$ is the trivial representation. Next we observe, that $\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y$ as well as $\mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2$ are invariant subspaces which are our candidates for the irreducible components, and we will only have to check irreducibility, which we will do at the end. At first we choose $(1,x,y,x^2,xy,y^2)$ as the order of our basis vectors in V. Hence we get for the adjusted φ with $\sigma_k \mapsto i\sigma_k$

$$\begin{aligned}\varphi(\alpha_1(i\sigma_1),\alpha_2(i\sigma_2),\alpha_3(i\sigma_3)).(a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2) =\\ = 1\cdot 0 +\\ + x\cdot(\alpha_1a_2+i\alpha_2a_2+i\alpha_3a_1)+\\ + y\cdot(\alpha_1a_1-i\alpha_2a_1-i\alpha_3a_2)+\\ + x^2\cdot(-2\alpha_1a_4+2i\alpha_2a_4+2i\alpha_3a_3)+\\ + xy\cdot(\alpha_1a_3+\alpha_1a_5+i\alpha_2a_3-i\alpha_2a_5)+\\ + y^2\cdot(-2\alpha_1a_4-2i\alpha_2a_4-2i\alpha_3a_5)\end{aligned}$$

and

$$\varphi(\alpha_1,\alpha_2,\alpha_3) \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\alpha_3 & \alpha_1+i\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1-i\alpha_2 & -i\alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i\alpha_3 & -2\alpha_1+2i\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1+i\alpha_2 & 0 & \alpha_1-i\alpha_2 \\ 0 & 0 & 0 & 0 & -2\alpha_1-2i\alpha_2 & -2i\alpha_3 \end{bmatrix}$$

Since φ is linear, it remains to show that it defines a Lie algebra homomorphism, i.e. that $\varphi([A,B]) = [\varphi(A),\varphi(B)]$: The multiplication in $\mathfrak{g} = \mathfrak{su}(2,\mathbb{C})$ goes

$$[\alpha_1(i\sigma_1)+\alpha_2(i\sigma_2)+\alpha_2(i\sigma_2),\alpha'_1(i\sigma_1)+\alpha'_2(i\sigma_2)+\alpha'_3(i\sigma_3)]$$

$$= \left[\begin{bmatrix} i\alpha_3 & i\alpha_1+\alpha_2 \\ i\alpha_1-\alpha_2 & -i\alpha_3 \end{bmatrix}, \begin{bmatrix} i\alpha'_3 & i\alpha'_1+\alpha'_2 \\ i\alpha'_1-\alpha'_2 & -i\alpha'_3 \end{bmatrix}\right]$$

$$= 2\cdot \begin{bmatrix} -i(\alpha_1\alpha'_2-\alpha_2\alpha'_1) & (\alpha_1\alpha'_3-\alpha_3\alpha'_1)-i(\alpha_2\alpha'_3-\alpha_3\alpha'_2) \\ -(\alpha_1\alpha'_3-\alpha_3\alpha'_1)-i(\alpha_2\alpha'_3-\alpha_3\alpha'_2) & i(\alpha_1\alpha'_2-\alpha_2\alpha'_1) \end{bmatrix}$$

$$= -2(\alpha_2\alpha'_3-\alpha_3\alpha'_2)(i\sigma_1)+2(\alpha_1\alpha'_3-\alpha_3\alpha'_1)(i\sigma_2)-2(\alpha_1\alpha'_2-\alpha_2\alpha'_1)(i\sigma_3)$$

We write $V = \underbrace{\mathbb{C}\cdot 1}_{V_0} \oplus \underbrace{\mathbb{C}x\oplus\mathbb{C}y}_{V_1} \oplus \underbrace{\mathbb{C}x^2\oplus\mathbb{C}xy\oplus\mathbb{C}y^2}_{V_2}$. Since the matrix of φ are diagonal blocks, and $\varphi|_{V_0}=0$ it is sufficient to show that $\varphi|_{V_1}$ and $\varphi|_{V_2}$ are homomorphisms. But we just calculated this property on V_1, which corresponds to the Lie algebra multiplication of the matrices in $\mathfrak{g}$ and it remains to show that

$$[\varphi|_{V_2}(\alpha_1,\alpha_2,\alpha_3),\varphi|_{V_2}(\alpha'_1,\alpha'_2,\alpha'_3)]$$

$$= \left[\begin{bmatrix} 2i\alpha_3 & -2\alpha_1+2i\alpha_2 & 0 \\ \alpha_1+i\alpha_2 & 0 & \alpha_1-i\alpha_2 \\ 0 & -2\alpha_1-2i\alpha_2 & -2i\alpha_3 \end{bmatrix}, \begin{bmatrix} 2i\alpha'_3 & -2\alpha'_1+2i\alpha'_2 & 0 \\ \alpha'_1+i\alpha'_2 & 0 & \alpha'_1-i\alpha'_2 \\ 0 & -2\alpha'_1-2i\alpha'_2 & -2i\alpha'_3 \end{bmatrix}\right]$$

$$= 2i \begin{bmatrix} -2(\alpha_1\alpha'_2-\alpha_2\alpha'_1) & 2(\alpha_1\alpha'_3-\alpha_3\alpha'_1) & 0 \\ \alpha_1\alpha'_3-\alpha_3\alpha'_1 & 0 & -(\alpha_1\alpha'_3-\alpha_3\alpha'_1) \\ 0 & -2(\alpha_1\alpha'_3-\alpha_3\alpha'_1) & 2(\alpha_1\alpha'_2-\alpha_2\alpha'_1) \end{bmatrix}$$

$$+ 2 \begin{bmatrix} 0 & 2(\alpha_2\alpha'_3-\alpha_3\alpha'_2) & 0 \\ -(\alpha_2\alpha'_3-\alpha_3\alpha'_2) & 0 & -(\alpha_2\alpha'_3-\alpha_3\alpha'_2) \\ 0 & 2(\alpha_2\alpha'_3-\alpha_3\alpha'_2) & 0 \end{bmatrix}$$

$$= \varphi|_{V_2}(-2(\alpha_2\alpha'_3-\alpha_3\alpha'_2),2(\alpha_1\alpha'_3-\alpha_3\alpha'_1),-2(\alpha_1\alpha'_2-\alpha_2\alpha'_1))$$

$$= \varphi|_{V_2}([(\alpha_1,\alpha_2,\alpha_3),(\alpha'_1,\alpha'_2,\alpha'_3)])$$

We set $\mathfrak{h} = \mathbb{C}H$ with $H := (0,0,-i) = (-i)\cdot(i\sigma_3) = \sigma_3$ which is an Abelian and hence nilpotent subalgebra of $\mathfrak{g}$. By the formula for multiplication we have

$$[(0,0,\alpha_3),(\alpha'_1,\alpha'_2,\alpha'_3)] = (2\alpha_3\alpha'_2,-2\alpha_3\alpha'_1,0)\in\mathfrak{h} \text{ only if } \alpha'_1=\alpha'_2=0$$

which means that $\mathfrak{h}$ is self-normalizing and hence the one-dimensional Cartan subalgebra of $\mathfrak{g}$.

Since $\varphi|_{V_0} = 0$ a vector of maximal height is $v_m = 0$.

Since $\varphi|_{V_1}(H).(a_1, a_2) = (a_1, -a_2)$ already is a basis of eigenvectors $x = (1,0), y = (0,1)$ to the eigenvalues ± 1 for the operation by H, the operation by the other two basis vectors of $\mathfrak{g}$ switch between those eigenspaces. We therefore change the basis for now and define

$$X := -\frac{1}{2}i(i\sigma_1) + \frac{1}{2}(i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y := -\frac{1}{2}i(i\sigma_1) - \frac{1}{2}(i\sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus

$$\varphi|_{V_1}(X).(a_1, a_2) = \varphi|_{V_1}\left(\left(-\frac{1}{2}i, +\frac{1}{2}, 0\right)\right)(a_1, a_2) = (a_2, 0)$$
$$\varphi|_{V_1}(Y).(a_1, a_2) = \varphi|_{V_1}\left(\left(-\frac{1}{2}i, -\frac{1}{2}, 0\right)\right)(a_1, a_2) = (0, a_1)$$
$$(0,1) \xrightarrow{X} (1,0) \xrightarrow{X} (0,0)$$
$$(1,0) \xrightarrow{Y} (0,1) \xrightarrow{Y} (0,0)$$

Let's take $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as the "ladder up" operator. Then we have $[H, X] = 2X$, i.e. $\lambda = 2$ is our positive root, and $v_m = (1,0)$ a vector of maximal weight $+1$, because - cp. section 7.1 in https://www.physicsforums.com/insights/journey-manifold-su2-part-ii/

$$H.v_m = \varphi|_{V_1}(H)(1,0) = +1 \cdot (1,0) \text{ diagonal}$$
$$X.v_m = \varphi|_{V_1}(X)(1,0) = (0,0) \text{ "ladder up"}$$
$$Y.v_m = \varphi|_{V_1}(Y)(1,0) = (0,1) \text{ "ladder down"}$$

This also shows, that $V_1 = \mathbb{C}x \oplus \mathbb{C}y$ is an irreducible component.

It remains to show the same for V_2. With the same settings as above we get $\varphi|_{V_2}(H) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and directly see the weights $-2, 0, 2$ with the eigenvectors $v_m = x^2 = (1,0,0), xy = (0,1,0), y^2 = (0,0,1)$.

$$\varphi|_{V_2}(X).(a_3, a_4, a_5) = \varphi|_{V_2}\left(\left(-\frac{1}{2}i, +\frac{1}{2}, 0\right)\right)(a_3, a_4, a_5) = (2ia_4, -ia_5, 0)$$
$$\varphi|_{V_2}(Y).(a_3, a_4, a_5) = \varphi|_{V_2}\left(\left(-\frac{1}{2}i, -\frac{1}{2}, 0\right)\right)(a_3, a_4, a_5) = (0, -ia_3, 2ia_4)$$
$$(0,0,1) \xrightarrow{X} (0,-i,0) \xrightarrow{X} (2,0,0) \xrightarrow{X} (0,0,0)$$
$$(1,0,0) \xrightarrow{Y} (0,-i,0) \xrightarrow{Y} (0,0,2) \xrightarrow{Y} (0,0,0)$$

which again shows the irreducibility, since we can jump from vector to vector along the entire $V_2 = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$ by the operation φ.

3. Let $(X\ ,\ ||\,.\,||)$ be a normed vector space. Prove that X is complete if and only if for each sequence with $\sum_{n=1}^{\infty} ||x_n|| < \infty$ the series $\sum_{n=1}^{\infty} x_n$ converges as well in X.

   Reason: Completeness criterion. (M)

   Solution: Assume X is complete and the sum of the normed sequence is finite. Then we have

$$||\sum_{n=1}^{\infty} x_n|| \leq \sum_{n=1}^{\infty} ||x_n|| < \infty$$

   and as X is complete, it converges even in X. Let on the other hand $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Then we set $a_n := x_{N(n+1)} - x_{N(n)}$ where we found the indices $N(n)$ in such a way, that $||x_m - x_k|| < \varepsilon_n := 2^{-n}$ for all $m, k > N(n)$. Since we have now

$$\sum_{n=1}^{\infty} ||a_n|| \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$

   the series $\sum_{n=1}^{\infty} a_n$ converges in X by our assumption. Hence

$$\sum_{n=1}^{\infty} a_n = \lim_{M\to\infty} \sum_{n<M} a_n = \lim_{M\to\infty} \left(-x_{N(1)} + x_{N(M)}\right) = -x_{N(1)} + \lim_{M\to\infty} x_{N(M)}$$

   and the limit of our sequence $(x_n)_{n\in\mathbb{N}}$ exists also in X and $(X\ ,\ ||\,.\,||)$ is complete.

4. Gauss' Divergence Theorem: $\iiint_V (\nabla F)\, dV = \iint_{\partial V} (F \cdot N) d(\partial V)$. See https://www.physicsforums.com/insights/pantheon-derivatives-part-v/

   (a) Let $B = B_1(0)$ the unit sphere in $\mathbb{R}^3$ and consider the vector field

$$F(x) = \begin{bmatrix} (x_2^4 + 2x_2^2x_3^2)x_1 \\ (x_3^4 + 2x_1^2x_3^2)x_2 \\ (x_1^4 + 2x_1^2x_2^2)x_3 \end{bmatrix}$$

   and calculate the integral $\int_{\partial B} F \cdot N \, d\mathbb{S}^2$

Reason: Gauss' Divergence Theorem. (E)

Solution: By Gauss' divergence theorem (the ball has a smooth boundary) we know that (N being the unit normal vector field)

$$\begin{aligned}
\int_{\partial B} F \cdot N \, d\mathbb{S}^2 &= \int_B \operatorname{div} F \, dB \\
&= \int_B \sum_{i=1}^{3} \frac{\partial}{\partial x_i} F_i(x) \, dB \\
&= \int_B (x_1^2 + x_2^2 + x_3^2)^2 \, dB \\
&= \int_B |x|^4 \, dB \\
&= \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\theta) r^4 \, d\theta \, d\phi \, dr \\
&= 2\pi \int_0^1 r^6 \, dr \int_0^{\pi} \sin(\theta) \, d\theta \\
&= \frac{4}{7}\pi
\end{aligned}$$

(b) Let $U \subseteq \mathbb{R}^n$ be open and $h \in C^1(U)$, $F \in C^1(U, \mathbb{R}^n)$. Show that on U we have

$$\operatorname{div}(hF) = h \operatorname{div} F + \nabla h \cdot F$$

Solution: This is basically the Leibniz rule, i.e. we have

$$\begin{aligned}
\operatorname{div}(hF) &= \sum (\frac{\partial}{\partial x_i})(hF_i) \\
&= \sum (\frac{\partial}{\partial x_i})(h) \cdot F_i + h \cdot \sum (\frac{\partial}{\partial x_i})(F_i) \\
&= \nabla h \cdot F + h \cdot \operatorname{div} F
\end{aligned}$$

(c) Let $B^n \subseteq \mathbb{R}^n$ be the closed unit ball and $f, g \in C^2(B^n)$. Show that with the unit normal vector field N

$$\int_{B^n} f \Delta g \, dB^n = -\int_{B^n} \nabla f \cdot \nabla g \; dB^n + \int_{\partial B^n} f \nabla g \cdot N \, d\mathbb{S}^{N-1}$$

Solution: We apply the previous formula and get

$$\begin{aligned}\int_{B^n} f\Delta g \, dB^n &= \int_{B^n} (\operatorname{div}(f\nabla g) - \nabla f \cdot \nabla g) \; dB^n \\ &= \int_{B^n} \operatorname{div}(f\nabla g) \, dB^n - \int_{B^n} \nabla f \cdot \nabla g \, dB^n \\ &= \int_{\partial B^n} f\nabla g \cdot N \, d\mathbb{S}^{n-1} - \int_{B^n} \nabla f \cdot \nabla g \, dB^n\end{aligned}$$

5. Let $f : (0,1) \longrightarrow \mathbb{R}$ be Lebesgue integrable and

$$Y := \{ (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1, x_2 \geq 0 \, , \, x_1 + x_2 \leq 1 \}$$

Show that for any α_1 , $\alpha_2 > 0$

$$\int_Y f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2} \, d\lambda(x_1,x_2) = \left[\int_0^1 f(u)u^{\alpha_1+\alpha_2+1} \, d\lambda(u)\right] \cdot \left[\int_0^1 v^{\alpha_1}(1-v)^{\alpha_2} \, d\lambda(v)\right]$$

Reason: Transformationtheorem. (T)

Solution: We define $\phi : (0,1)^2 \longrightarrow \mathbb{R}^2$ by $\phi(u,v) = (vu, (1-v)u)$. Now $\operatorname{im}\phi = \{ (x_1,x_2) \in \mathbb{R}^2 \,|\, x_1, x_2 > 0 \, , \, x_1 + x_2 < 1 \} = Y^\circ$ the inner of Y and ϕ is bijective with $\phi^{-1}(x_1,x_2) = \left(x_1 + x_2, \dfrac{x_1}{x_1+x_2}\right)$. We further have $\det D\phi(u,v) = -u$ and note that $Y - Y^\circ$ is a nullset with respect to $\lambda(Y)$. Hence

$$\begin{aligned}&\int_Y f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2} \, d\lambda(x_1,x_2) \\ =& \int_Y f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2} \, d\lambda(x_1,x_2) \\ =& \int_{(0,1)^2} f(vu + (1-v)u) \, (vu)^{\alpha_1} \, ((1-v)u)^{\alpha_2} \, u \; d\lambda(u,v) \\ =& \int_{(0,1)^2} f(u) \, u^{\alpha_1+\alpha_2+1} \, v^{\alpha_1}(1-v)^{\alpha_2} \; d\lambda(u,v) \\ \stackrel{Fubini}{=}& \left[\int_0^1 f(u)u^{\alpha_1+\alpha_2+1} \, d\lambda(u)\right] \cdot \left[\int_0^1 v^{\alpha_1}(1-v)^{\alpha_2} \, d\lambda(v)\right]\end{aligned}$$

6. Finite Groups. **Reason:** Basics. (E)

(a) Let $U \subsetneq G$ be a proper subgroup of a finite group.
Show that $\bigcup_{g\in G} gUg^{-1} \subsetneq G$ is a proper subset.

(b) Let $G \neq \{1\}$ be a finite group which operates transitive on X which has at least two elements $|X| > 1$. Transitive means all elements of X can be reached by the group operation from a single $x \in X$. Show that there is a group element $g \in G$ such that $g.x \neq x$ for all $x \in X$.

Solution:

(a) If U is a normal subgroup, we are done. So let us assume that $U \subsetneq G$ isn't normal. The conjugation in G gives us an operation on the powerset $\mathcal{P}G$ of G by:

$$\begin{aligned} G \times \mathcal{P}G &\longrightarrow \mathcal{P}G \\ (g, M) &\longmapsto g.M := gMg^{-1} = \{\, gmg^{-1} \mid m \in M \,\} \end{aligned}$$

The normalizer $N_G(U) = \{\, gUg^{-1} \mid gUg^{-1} \subseteq U \,\}$ of U in G is the largest subgroup of G which contains U as a normal subgroup. By the above operation we get

$$|G| = |N_G(U)| \cdot |G.U|$$

and therefore

$$\begin{aligned} |\bigcup_{g\in G} gUg^{-1}| = |\bigcup_{H\in G.U} H| &\overset{(*)}{<} \bigcup_{H\in G.U} |H| = \bigcup_{H\in G.U} |U| \\ &= |G.U| \cdot |U| = \frac{|G|}{|N_G(U)|} \cdot |U| \leq \frac{|G|}{|U|} \cdot |U| = |G| \end{aligned}$$

The inequality in $(*)$ is proper, because $1 \in H$ for all H and the union contains at least two sets, since U isn't normal in G.

(b) For a given $x \in X$ we consider its stabilizer

$$\operatorname{Stab}_G(x) = \{\, g \in G \mid g.x = x \,\}$$

Since G operates transitive on X and $|X| > 1$ the stabilizer of x is a proper subgroup of G:

$$\operatorname{Stab}_G(x) = G \Longrightarrow g.x = x \neq y \in X$$

so either G would not operate transitive or X would not contain two different elements. Furthermore are all stabilizers $\operatorname{Stab}_G(y)$ conjugates of $\operatorname{Stab}_G(x)$: Let $g.x = y$. Then $ghg^{-1}.y = gh.x = g.x = y$, i.e. conjugation by g maps $\operatorname{Stab}_G(x)$ isomorph on $\operatorname{Stab}_G(y)$. By the previous part we have

$$\bigcup_{g\in G} g \operatorname{Stab}_G(x) g^{-1} = \bigcup_{y\in X} \operatorname{Stab}_G(y) \neq G$$

Hence there is a $g \in G$ with $g \notin \operatorname{Stab}_G(y)$ for all $y \in X$, which means g doesn't fix any element of X, which we had to prove.

7. Let

$$\begin{aligned} O_n(\mathbb{R}) &= \{ A \in \mathbb{M}(n,\mathbb{R}) \,|\, \langle Av, Aw \rangle = \langle v, w \rangle \text{ for all } v, w \in \mathbb{R}^n \} \\ &= \{ A \in \mathbb{M}(n,\mathbb{R}) \,|\, A^\tau A = AA^\tau = 1 \} \end{aligned}$$

be the orthogonal group of $n \times n$ matrices which operate per matrix multiplication on $\mathbb{R}^n$ $(n \in N)$.

(a) Determine the orbit of $x \in \mathbb{R}^n$ under $O_n(\mathbb{R})$.

(b) Determine the stabilizer $\operatorname{Stab}_x(O_n(\mathbb{R})) = \{ A \in O_n(\mathbb{R}) \,|\, A.x = x \}$ of $x = (0, 0, \ldots, 1) \in \mathbb{R}^n$ in $O_n(\mathbb{R})$.

(c) Determine a bijection $\mathbb{S}^{n-1} \stackrel{1:1}{\longleftrightarrow} O_n(\mathbb{R})/O_{n-1}(\mathbb{R})$ between the unit sphere in $\mathbb{R}^n$ and the factor of two consecutive orthogonal groups.

Reason: For physicists. (E)

Solution: From the defining relation we get

$$\langle Av, Aw \rangle = (Av)^\tau (Aw) = v^\tau A^\tau A w = v^\tau (A^\tau A) w = v^\tau w = \langle v, w \rangle$$

that every $A \in O_n(\mathbb{R})$ is isometric and $||Ax|| = ||x||$. Thus the orbit of $x \in O_n(\mathbb{R})$ is

$$O_n(\mathbb{R}).x \subseteq \{ v \in O_n(\mathbb{R}) \,|\, ||v|| = ||x|| \} = ||x|| \cdot \mathbb{S}^{n-1}$$

For $n = 1$ we are done. The other inclusion " $\supseteq$ " also holds, as $O_n(\mathbb{R})$ operates transitive on $\mathbb{S}^{n-1}$: For $v \neq w$ in $||x|| \cdot \mathbb{S}^{n-1}$, i.e. $||v|| = ||x|| = ||w||$ we consider the plane $\operatorname{span}_{\mathbb{R}}\{ v, w \}$, $n > 1$. With a rotation axis in its origin and perpendicular to the plane, we can rotate v into w by an appropriate element of $O_n(\mathbb{R})$ and hence v, w are in the same orbit, which is the orbit of x - just select $w = x$.

In order to get a fixed point $A.x = x$ with $x = (0, 0, \ldots, 1)$ the matrix A has to be of the form
$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix}$$
Since matrix multiplication is blockwise, we get $(A')^\tau A' = A'(A')^\tau = 1$ and thus $A' \in O_{n-1}(\mathbb{R})$ and

$$\operatorname{Stab}_{(0,0,\ldots,1)}(O_n(\mathbb{R})) = O_{n-1}(\mathbb{R}) \subseteq O_n(\mathbb{R})$$

by the embedding $A' \longmapsto A$ as above.

Now we know, that $O_n(\mathbb{R})$ operates transitive on $\mathbb{S}^{n-1}$ and the quotient of two consecutive orthogonal groups is the quotient of $O_n(\mathbb{R})$ by the stabilizer $\operatorname{Stab}_{(0,0,\ldots,1)}(O_n(\mathbb{R}))$. By the formula for orbits

$$G/\operatorname{Stab}_x(G) \cong G.x$$

we get

$$\begin{aligned} O_n(\mathbb{R})/O_{n-1}(\mathbb{R}) &\cong O_n(\mathbb{R})/\operatorname{Stab}_{(0,0,\ldots,1)}(O_n(\mathbb{R})) \\ &\cong O_n(\mathbb{R}).(0,0,\ldots,1) \\ &= ||(0,0,\ldots,1)|| \cdot \mathbb{S}^{n-1} \\ &= \mathbb{S}^{n-1} \end{aligned}$$

8. We define an equivalence relation on the topological two-dimensional unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ by $x \sim y \Longleftrightarrow x \in \{\pm y\}$ and the projection $q : \mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$. Furthermore we consider the homeomorphism $\tau : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ defined by $\tau(x) = -x$. Note that for $A \subseteq \mathbb{S}^2$ we have $q^{-1}(q(A)) = A \cup \tau(A)$. Show that
   (a) q is open and closed.
   (b) $\mathbb{S}^2/\sim$ is compact, i.e. Hausdorff and covering compact.
   (c) Let $U_x = \{ y \in \mathbb{S}^2 : ||y - x|| < 1 \}$ be an open neighborhood of $x \in \mathbb{S}^2$. Show that $U_x \cap U_{-x} = \emptyset$, $U_{-x} = \tau(U_x)$, $q(U_x) = q(U_{-x})$ and $q|_{U_x}$ is injective. Conclude that q is a covering.

Reason: Standard Sphere. (E)

Solution:

(a) Let $O \subseteq \mathbb{S}^2$ be open. Then $O \cup \tau(O) = q^{-1}(q(O))$ is open and by definition of the quotient topology $q(O) \subseteq \mathbb{S}^2/\sim$ is open and so is q. Let $B \subseteq \mathbb{S}^2$ be closed. Then by the same argument, $q(B)$ is closed and so is q.

(b) $\mathbb{S}^2/\sim$ is covering-compact as $\mathbb{S}^2$ is covering-compact, and the continuous function q is surjective. Furthermore q is closed by the previous part, so $\mathbb{S}^2/\sim$ is Hausdorff because $\mathbb{S}^2$ is.

(c) We would have $2 = ||x - (-x)|| \leq ||x - y|| + ||y - (-x)|| < 2$, so $U_x \cap U_{-x} = \emptyset$. We also have $-y \in U_x$ if and only if $y \in U_{-x}$ and so $U_{-x} = \tau(U_x)$. From $q \circ \tau = q$ we thus have $q(U_x) = q(\tau(U_x)) = q(U_{-x})$.

Now let $y, y' \in U_x$ such that $q(y) = q(y')$. Then $y' = \pm y$. As from $y' = -y \in U_x$ we would get $y \in U_{-x} \cap U_x = \emptyset$ which is impossible. Hence $y' = y$ and the restriction of q on U_x is injective.

By the previous we have that $q : \mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$ is a continuous, open and closed, surjection. So we may set $U_{q(x)} := q(U_x)$ as open neighborhood of an element in $\mathbb{S}^2/\sim$ with $q^{-1}(U_{q(x)}) = U_x \dot{\cup} U_{-x}$, a disjoint union of two open sets in $\mathbb{S}^2$. We also know that q is injective on U_x, hence maps U_x homeomorph onto $U_{q(x)}$ and equally with U_{-x}. Thus all properties of a covering map are fulfilled.

9. A function $|\,.\,| : \mathbb{F} \longrightarrow \mathbb{R}_{\geq 0}$ on a field $\mathbb{F}$ is called a value function if

$$|x| = 0 \Longleftrightarrow x = 0$$
$$|xy| = |x|\,|y|$$
$$|x + y| \leq |x| + |y|$$

It is called Archimedean, if for any two elements a, b $(a \neq 0)$ there is a natural number n such that $|na| > |b|$. We consider the rational numbers. The usual absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is Archimedean, whereas the trivial value

$$|x|_0 = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

is not.
Determine all non-trivial and non-Archimedean value functions on $\mathbb{Q}$.

Reason: Ostrowski's Theorem. (D)

Solution: Since $|\,.\,|$ is non-Archimedean, there are elements a,b with $|n| < \frac{|b|}{|a|}$ for all $n \in \mathbb{N}$. If $|n| > 1$ for a natural number, then $|n^k| = |n|^k$ goes to infinity and cannot be bounded. Thus $|n| \leq 1$ for all $n \in \mathbb{N}$. Let's assume $|n| = 1$ for all $n \in \mathbb{N}$. Then for $a = \frac{m}{n}$ we get $1 = |m| = |an| = |a|\,|n| = |a|$ and the value function is trivial. Thus there is a smallest n_0 by its natural ordering with $|n_0| < 1$. Let's assume $n_0 = ab$. This means that either $|a| < 1$ or $|b| < 1$ and by minimality of n_0 we have $n_0 = a$ or $n_0 = b$ and thus $n_0 =: p$ is prime.

We next show that $|a{+}b| \leq \max\{\,|a|,|b|\,\}$. This is equivalent to $|a{+}1| \leq \max\{\,|x|,1\,\}$ which can be seen by division by b and $|1| = 1$.

$$\begin{aligned} |a+1|^m &= \left|\sum_{k=0}^{m}\binom{m}{k}a^k\right| \leq \sum_{k=0}^{m}\left|\binom{m}{k}\right|\,|a|^k \leq \sum_{k=0}^{m}|a|^k \\ &\leq (m+1)\max\{\,|a|^m,1\,\} \end{aligned}$$

hence $|a+1| \leq \sqrt[m]{m+1}\,\max\{\,|a|,1\,\}$. Since $\lim_{m\to\infty}\sqrt[m]{m+1} = 1$ we have $|a+1| \leq \max\{\,|a|,1\,\}$.

Let $m = kp + r$ with $p \nmid m$ and $r \in \{\,1,\ldots,p-1\,\}$. By minimality of p we have $|r| = 1$, and $|kp| = |k|\,|p| \leq |p| < 1$, so

$$|m| = |kp + r| \leq \max\{\,|kp|,|r|\,\} = 1$$

Now let $|a| < |b|$. Then we get

$$|a| < |b| = |(a{+}b){-}a| \leq \max\{\,|a{+}b|,|a|\,\} = |a{+}b| \leq \max\{\,|a|,|b|\,\} = |b|$$

and $|a+b| = \max\{\,|a|,|b|\,\}$.

So any natural number m which is coprime to p has $|m| = 1$ and all others are of the form $m = p^r m'$ with $|m| = |p|^r|m'| = |p|^r$. If we set $\alpha := \dfrac{\log p^{-1}}{|\log p|}$ we get $|p|^\alpha = p^{-1}$ and $|m|^\alpha = |p|^{r\alpha} = p^{-r}$.

For $m = p^r \cdot m'$, $n = p^s \cdot n'$ with $(m',p) = (n',p) = 1$ we therefore have

$$\left|\frac{m}{n}\right| = \begin{cases} 0 & \text{if } m = 0 \\ p^{-r+s} & \text{if } m \neq 0 \end{cases}$$

b which is the p-adic absolute value (norm) of $\mathbb{Q}$.

The topological completion of $\mathbb{Q}$ with respect to the p-adic norm is called the field of p-adic numbers $\mathbb{Q}_p$. It is a field with prime field $\mathbb{Q}$ and thus is of characteristic zero. Its algebraic closure is of infinite degree. So $\mathbb{Q}_p$ has infinitely many inequivalent algebraic extensions.

10. For a set X let

$$\mathcal{B}(X) = \{\, f \,:\, X \longrightarrow \mathbb{R} \,:\, \sup_{x\in X}\{\, |f(x)| \,\} =: ||f||_\infty < \infty \,\}$$

be the space of all bounded functions on X. We define a metric on $\mathcal{B}(X)$ by $d(f,g) = ||f-g||_\infty$.

(a) Show that $(\mathcal{B}(X)\,,\, d)$ is complete.

(b) If (X,d) is a metric space and $a \in X$. Prove that the function

$$\phi_a \,:\, X \longrightarrow \mathcal{B}(X)\,,\; \phi_a(x) = d(x,.) - d(a,.)$$

is an isometry of X in $\mathcal{B}(X)$.

(c) Show that the closure of $\operatorname{im}(\phi_a)$ is a completion of $X \sim \phi_a(X)$.

Reason: Functional Analysis Basics. (E)

Solution:

(a) Let (f_n) be a Cauchy sequence in $\mathcal{B}(X)$, that is for any $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n, m > N_\varepsilon$

$$||f_n - f_m||_\infty < \varepsilon$$

Thus we also have $|f_n(x) - f_m(x)| < \varepsilon$ and $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence, which converges to $\lim_{n\to\infty} f_n(x) =: f(x)$. For $\varepsilon = 1$ and $N := N_1 + 1$ we thus have

$$|f_n(x)| < |f_{N+1}(x)| + 1 \text{ for all } n > N \;,\; x \in X$$

and thus $|f(x)| < |f_{N+1}(x)| + 1 < ||f_{N+1}||_\infty + 1$ which shows $f \in \mathcal{B}(X)$. Now for $m \to \infty$ we get from $|f_n(x) - f_m(x)| < \varepsilon$ that $|f_n(x) - f(x)| \leq \varepsilon$ and $||f_n - f||_\infty \leq \varepsilon$ so (f_n) converges to f in $(\mathcal{B}(X)\,,\, ||\,.\,||_\infty)$.

(b) Since $||\phi_a(x)||_\infty = \sup_{y\in X} |d(x,y) - d(a,y)| \leq d(x,a) < \infty$ we have $\phi_a(x) \in \mathcal{B}(X)$.
From the triangle inequality we get $|d(x,z) - d(y,z)| \leq d(x,y)$

where equality holds for $z = x$, and thus $\sup_{z\in X} |d(x,z) - d(y,z)| = d(x,y)$. Therefore is

$$
\begin{aligned}
||\phi_a(x) - \phi_a(y)||_\infty &= \sup_{z\in X} |\phi_a(x)(z) - \phi_a(y)(z)| \\
&= \sup_{z\in X} |d(x,z) - d(a,z) - d(y,z) + d(a,z)| \\
&= \sup_{z\in X} |d(x,z) - d(y,z)| \\
&= d(x,y)
\end{aligned}
$$

and ϕ_a is an isometry for any (fixed) $a \in X$.

(c) A subset $N \subseteq M$ of a complete metric space is complete if and only if it is closed. So $\overline{N}$ is a completion of N .

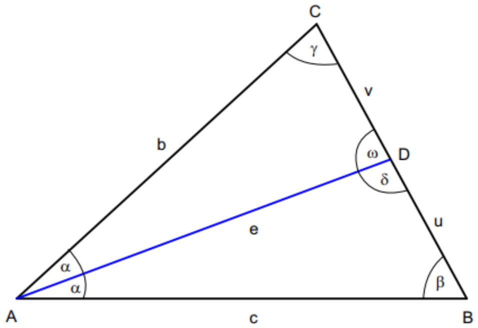


Figure 1: Triangle with bisector

6 July-B 2018

1. The angle bisector in a given triangle $\triangle\, ABC$ of the angle $\alpha = \sphericalangle BAC$ at A intersects the side $\overline{BC}$ at the point D. We have the information:

$$\overline{BD} \cdot \overline{CD} = \overline{AD}^{\,2} \tag{8}$$
$$\sphericalangle ADB = 45^\circ \tag{9}$$

   (a) Determine the inner angles of $\triangle\, ABC$
   (b) Determine the precise ratio at which D divides $\overline{BC}$

   Reason: Triangle practice.

   Solution: We are given $\delta = \sphericalangle ADB = 45^\circ$ and $\omega = \sphericalangle ADC = 135^\circ$, and asked for

$$\alpha' = 2\alpha = \sphericalangle BAC\ ,\ \beta = \sphericalangle ABC = 135^\circ - \alpha\ ,\ \gamma = \sphericalangle BCA = 45^\circ - \alpha$$

   The law of sines for $\triangle\, ADB$ and $\triangle\, ADC$ reads

$$\frac{u}{\sin\alpha} = \frac{e}{\sin\beta}\ ,\ \frac{v}{\sin\alpha} = \frac{e}{\sin\gamma}$$

   and we get with the condition $u \cdot v = e^2$

$$e^2 = \frac{u \cdot \sin\beta}{\sin\alpha} \cdot \frac{v \cdot \sin\gamma}{\sin\alpha} = e^2 \cdot \frac{\sin\beta \sin\gamma}{\sin^2\alpha}$$

   or

$$\sin^2\alpha = \sin(135^\circ - \alpha)\sin(45^\circ - \alpha) = \frac{1}{2}(\cos^2\alpha - \sin^2\alpha) = \frac{1}{2} - \sin^2\alpha$$

and $\alpha = \arcsin \frac{1}{2} = 30^\circ$, $\alpha' = 60^\circ$, $\beta = 105^\circ$, $\gamma = 15^\circ$.

Next we consider the law of cosine for $\triangle ABD$ and $\triangle ADC$ and get with our condition $uv = e^2$

$$\begin{aligned} c^2 &= u^2 + e^2 - \sqrt{2}ue = u(u + v - \sqrt{2uv}) \\ b^2 &= v^2 + e^2 + \sqrt{2}ve = v(u + v + \sqrt{2uv}) \end{aligned}$$

and the theorem of angle bisectors says $cv = bu$. Thus

$$c^2v^2 = uv^2(u + v - \sqrt{2uv}) = u^2v(u + v + \sqrt{2uv}) = b^2u^2$$

from which we get with $r = \frac{u}{v}$

$$\begin{aligned} v(u + v - \sqrt{2uv}) &= u(u + v + \sqrt{2uv}) \\ r^{-1} - \sqrt{2r^{-1}} &= r + \sqrt{2r} \\ 1 - \sqrt{2r} &= r^2 + r\sqrt{2r} \\ (1 - r^2)^2 &= 2r(1 + r)^2 \\ 0 &= r^2 - 4r + 1 \\ \frac{u}{v} &= r \in \left\{ 2 + \sqrt{3} \, , \frac{1}{2 + \sqrt{3}} \right\} \end{aligned}$$

From the first part of the question we have

$$\frac{u}{v} = \frac{\sin \gamma}{\sin \beta} = \frac{\sin 15^\circ}{\sin 105^\circ} < 1, \text{ i.e. } \frac{u}{v} = \frac{1}{2 + \sqrt{3}}$$

2. At the cash desk of a shopping center five friends (Diana, Ike, Jessica, Stan, Valery) are standing in a row. They are all different in age (26, 27, 30, 33 and 35 years) and would like to buy all different tops (shirt, polo shirt, pullover, sweatshirt and T-shirt) for themselves. The tops are all different colors (blue, yellow, green, red and black) and different sizes (XS, S, M, L and XL).

   Find out who is where, how old and what top to buy in which color and size. The positions in the queue can be seen from the cashier, i.e. "front" or "the first person" is right at the cash register. There are no other people in the queue and the cashier is to be ignored.

   (a) 1. Diana, who wants to buy a top in size XL, stands further ahead than the person who wants to buy a black top.

   (b) 2. Jessica stands in front of the person who wants to buy a polo shirt.

(c) 3. The second person in the queue wants to buy a yellow top.

(d) 4. The t-shirt is not red.

(e) 5. Stan wants to buy a sweatshirt. The person standing in front of him is older than the person standing directly behind him.

(f) 6. Ike needs a top in size L.

(g) 7. The last person in the queue is 30 years old.

(h) 8. The oldest person wants to buy the top in the smallest size.

(i) 9. The person standing directly behind Valery wants to buy a red top that is larger than size S.

(j) 10. The youngest person wants to buy a yellow top.

(k) 11. Jessica wants to buy a shirt.

(l) 12. The third person in the queue wants to buy a top in size M.

(m) 13. The polo shirt is red, yellow or green.

Reason: Logic.

Solution:

- 3-7-12: (25),(42),(53)
- 10: +(22)
- 1-2-5-6-9: +(15),(55)
- 1-2-5-9: +(12),(32)
- 1-2-9: +(11),(51)
- 8: +(24),(52),(54)
- 5: +(21),(23)
- 9: +(14),(45)
- 2-11: +(13),(33),(34)
- 4: +(31),(35)
- 13: +(44)
- 1: +(41),(43)

Diana, Ike, Jessica, Stan, Valery

shirt, polo shirt, pullover, sweatshirt and T-shirt

Position	1	2	3	4	5
Person	Dana	Sören	Jessica	Valerie	Ingo
Alter	33	26	27	35	30
Oberteil	T-Shirt	Sweatshirt	Hemd	Poloshirt	Pullover
Farbe	Blau	Gelb	Schwarz	Grün	Rot
Größe	XL	S	M	XS	L

Figure 2: At the Cash Desk

3. Compute the arc length $\mathcal{L}$ of the cycloid

$$\gamma \,:\, \mathbb{R} \longrightarrow \mathbb{R}^2\,,\; \gamma(t) = (t - \sin(t), 1 - \cos(t))$$

between two neighboring singularities.

Reason: Training for physicists.

Solution: The singularities $\gamma'(t) = 0$ are $2\pi k$ $(k \in \mathbb{Z})$.

$$\begin{aligned}
\mathcal{L} &= \int_{2\pi k}^{2\pi(k+1)} ||\dot{\gamma}(t)||\,dt = \int_{2\pi k}^{2\pi(k+1)} \left|\left|\begin{pmatrix} 1-\cos(t) \\ \sin(t)\end{pmatrix}\right|\right| \,dt \\
&= \sqrt{2}\int_{2\pi k}^{2\pi(k+1)} \sqrt{1-\cos(t)}\,dt \\
&= \sqrt{2}\int_{2\pi k}^{2\pi(k+1)} \sqrt{1-\left(\cos^2\left(\frac{t}{2}\right)-\sin^2\left(\frac{t}{2}\right)\right)}\,dt \\
&= \sqrt{2}\int_{2\pi}^{2\pi(k+1)} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt = 2\int_{2\pi}^{2\pi(k+1)} \left|\sin\left(\frac{t}{2}\right)\right|\,dt \\
&= 4\int_0^{\pi} |\sin(x)|\,dx = 4\,(-\cos(\pi)+\cos(0)) \\
&= 8
\end{aligned}$$

7 July-I 2018

1. Let's consider complex functions in one variable and especially the involutions

$$\mathcal{I} = \{\, z \stackrel{p}{\mapsto} z \,,\; z \stackrel{q}{\mapsto} -z \,,\; z \stackrel{r}{\mapsto} z^{-1} \,,\; z \stackrel{s}{\mapsto} -z^{-1} \,\}$$

We also consider the two functions

$$\mathcal{J} = \{\, z \stackrel{u}{\longmapsto} \frac{1}{2}(-1+i\sqrt{3})z \,,\; z \stackrel{v}{\longmapsto} -\frac{1}{2}(1+i\sqrt{3})z \,\}$$

and the set $\mathcal{F}$ of functions which we get, if we combine any of them: $\mathcal{F} = \langle \mathcal{I}, \mathcal{J} \rangle$ by consecutive applications. We now define for $\mathcal{K} \in \{\mathcal{I}, \mathcal{J}\}$ a relation on $\mathcal{F}$ by

$$f(z) \sim_{\mathcal{K}} g(z) \;:\Longleftrightarrow\; (\forall\, h_1 \in \mathcal{K})\,(\exists\, h_2 \in \mathcal{K})\,:\, f(h_1(z)) = g(h_2(z))$$

(a) Show that $\sim_{\mathcal{K}}$ defines an equivalence relation.

(b) Show that $\mathcal{F}/\sim_{\mathcal{J}}$ admits a group structure on its equivalence classes by consecutive application.

(c) Show that $\mathcal{F}/\sim_{\mathcal{I}}$ does not admit a group structure on its equivalence classes by consecutive applications.

Reason: Normal subgroups versus ordinary subgroups.

Solution: The relation is that of subgroups in a group and we have

$$D_6 \cong \mathcal{F} \cong \mathcal{I} \ltimes \mathcal{J} \cong V_4 \ltimes \mathbb{Z}_3 \cong D_3 \times \mathbb{Z}_2$$

with $\varphi : V_4 \longrightarrow \operatorname{Aut}(\mathbb{Z}_3)$, $\varphi(t)(w)(z) := (twt)(z) = t(w(t(z)))$. As $\mathcal{I} \ntrianglelefteq \mathcal{F}$ isn't a normal subgroup, $\mathcal{F}/\sim_{\mathcal{I}}$ doesn't carry a group structure for the same reason as $\sim_{\mathcal{J}}$ does.

$$D_6 = \langle (uq), r \mid (uq)^6 = r^2 = r(uq)r(uq) = 1 \rangle$$

2. We consider the vector field $X : \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $X(p) := \left(p, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

   (a) Compute the derivative $d\phi : T\mathbb{R}^2 \longrightarrow T\mathbb{R}^3$ of the stereographic projection to the north pole, i.e. plane to sphere with $\phi(0,0) = (0,0,-1)$, and describe the tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$. Show that position vectors and tangent vectors are orthogonal.

   (b) Compute the vector field $d\phi(X)$ on $\mathbb{S}^2$. How is it related to the curves $\gamma(t) = \phi(t, y_0)$?

   (c) Is $d\phi(X)$ a continuous vector field on $\mathbb{S}^2$ without zeros?

Reason: Training for physicists.

Solution:

(a) The stereographic projection to the north pole is given by

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\phi(x,y) = \frac{1}{x^2+y^2+1} \begin{bmatrix} 2x \\ 2y \\ x^2+y^2-1 \end{bmatrix}$$

from which we get

$$d_{(x,y)}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = \frac{1}{(x^2+y^2+1)^2}\begin{bmatrix} 2(1-x^2+y^2) & -4xy \\ -4xy & 2(1+x^2-y^2) \\ 4x & 4y \end{bmatrix}$$

and $d\phi \,:\, T\mathbb{R}^2 \longrightarrow T\mathbb{R}^3\,,\,(p,v) \longmapsto (\phi(p), d_p\phi(v))\,.$ The tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$ is given by the image of the $\phi-$parameterized set $\phi(\mathbb{S}^2)$ under $d\phi$. As expected is

$$\phi(p) \perp d_p(\phi)(v)$$

$$\begin{aligned}
&(1+r^2+s^2)^3 \cdot \left\langle \phi(r,s), d_{(r,s)}(\phi)(v_1,v_2)\right\rangle \\
&= \left\langle \begin{bmatrix} 2r \\ 2s \\ r^2+s^2-1 \end{bmatrix}, \begin{bmatrix} 2(1-r^2+s^2) & -4rs \\ -4rs & 2(1+r^2-s^2) \\ 4r & 4s \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 2r \\ 2s \\ r^2+s^2-1 \end{bmatrix}, \begin{bmatrix} 2(1-r^2+s^2)v_1 - 4rsv_2 \\ -4rsv_1 + 2(1+r^2-s^2)v_2 \\ 4rv_1 + 4sv_2 \end{bmatrix} \right\rangle \\
&= 4rv_1 - 4r^3v_1 + 4rs^2v_1 - 8r^2sv_2 - 8rs^2v_1 + 4sv_2 + 4r^2sv_2 \\
&- 4s^3v_2 + 4r^3v_1 + 4r^2sv_2 + 4rs^2v_1 + 4s^3v_2 - 4rv_1 - 4sv_2 \\
&= 0
\end{aligned}$$

(b) Since $X(p) = (1,0)$ we have

$$d_p\phi(X) = \left.\frac{\partial \phi}{\partial x}\right|_p (X) = \frac{1}{(r^2+s^2+1)^2}\begin{bmatrix} 2(1-r^2+s^2) \\ -4rs \\ 4r \end{bmatrix}$$

at position $p = (r,s)\,,$ the unit tangent vector field along the curve $\gamma \,:\, \mathbb{R} \longrightarrow \mathbb{S}^2\,,\, \gamma(r) = \phi(r,s) = (r^2+s^2+1)^{-1}(2r, 2s, r^2+s^2-1)$ for a certain $s \in \mathbb{R}\,.$

(c) The vector field $d\phi(X) = (p, d_p\phi(X))$ is defined on the image of ϕ, namely $\mathbb{S}^2 - \{N\}$ where the pole N is missing. It's the point at infinity. $d\phi(X)$ is continuous on its domain without zeros, but it can be continuously extended over the north pole. However, here we get

$$\lim_{|p|\to\infty} d_p\phi(X) = 0$$

so this continuous extension has a zero at the pole.

3. (On the occasion of the centenary of Emmy Noether's theorem.)

   The action on a classical particle is the integral of an orbit $\gamma\,:\,t \rightarrow \gamma(t)$

$$S(\gamma)=S(x(t))=\int \mathcal{L}(t,x,\dot{x})\,dt$$

   over the Lagrange function $\mathcal{L}$, which describes the system considered. Now we consider smooth coordinate transformations

$$\begin{aligned} x \longmapsto x^* &:= x+\varepsilon\psi(t,x,\dot{x})+O(\varepsilon^2)\\ t \longmapsto t^* &:= t+\varepsilon\varphi(t,x,\dot{x})+O(\varepsilon^2)\end{aligned}$$

   and we compare

$$S=S(x(t))=\int \mathcal{L}(t,x,\dot{x})\,dt \text{ and } S^*=S(x^*(t^*))=\int \mathcal{L}(t^*,x^*,\dot{x}^*)\,dt^*$$

   Since the functional S determines the law of motion of the particle,

$$S=S^*$$

   means, that the action on this particle is unchanged, i.e. invariant under these transformations, and especially

$$\frac{\partial S}{\partial \varepsilon}=0 \quad \text{resp.} \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\mathcal{L}\left(t^*,x^*,\dot{x}^*\right)\cdot\frac{dt^*}{dt}\right)=0 \qquad (10)$$

   Emmy Noether showed exactly hundred years ago, that under these circumstances (invariance), there is a conserved quantity Q. Q is called the Noether charge.

$$S=S^* \Longrightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\mathcal{L}\left(t^*,x^*,\dot{x}^*\right)\cdot\frac{dt^*}{dt}\right)=0 \Longrightarrow \frac{d}{dt}Q(t,x,\dot{x})=0$$

   with

$$Q=Q(t,x,\dot{x}):=\sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\,\psi_i+\left(\mathcal{L}-\sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\,\dot{x}_i\right)\varphi= \text{ constant}$$

   The general way to proceed is:

   (a) Determine the functions ψ,φ, i.e. the transformations, which are considered.

   (b) Check the symmetry by equation (10).

(c) If the symmetry condition holds, then compute the conservation quantity Q with $\mathcal{L}, \psi, \varphi$.

Example: Given a particle of mass m in the potential $U(\vec{r}) = \dfrac{U_0}{\vec{r}^{\,2}}$ with a constant U_0. At time $t = 0$ the particle is at $\vec{r}_0$ with velocity $\dot{\vec{r}}_0$.

Hint: The Lagrange function with $\vec{r} = (x, y, z, t) = (x_1, x_2, x_3, t)$ of this problem is

$$\mathcal{L} = T - U = \frac{m}{2}\,\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}}$$

(a) Give a reason why the energy of the particle is conserved, and what is its energy?

(b) Consider the following transformations with infinitesimal ε

$$\vec{r} \longmapsto \vec{r}^{\,*} = (1+\varepsilon)\,\vec{r}\ ,\ t \longmapsto t^* = (1+\varepsilon)^2\,t$$

and verify the condition (10) to E. Noether's theorem.

(c) Compute the corresponding Noether charge Q and evaluate Q for $t = 0$.

Reason: For physicists. Centenary.

Solution:

(a) i. Energy is *homogeneous in time*, so we chose $\psi_i = 0, \varphi = 1$

ii. and check equation (10) by

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\mathcal{L}^* \cdot \frac{d}{dt}\,(t+\varepsilon)\right) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\mathcal{L}^* \cdot 1) = 0$$

since $\mathcal{L}^*$ doesn't depend on t^* and thus not on ε, and calculate

iii. the Noether charge as

$$\begin{aligned} Q(t,x,\dot{x}) &= \mathcal{L} - \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\,\dot{x}_i \\ &= T - U - \frac{m}{2}\left(\frac{\partial}{\partial \dot{x}_i}\left(\sum_{i=1}^{3} \dot{x}_i^2\right)\dot{x}_i\right) \\ &= \frac{m}{2}\,\dot{\vec{r}}^{\,2} - U - m\,\dot{\vec{r}}^{\,2} \\ &= -T - U \\ &= -E \\ &= -\frac{m}{2}\,\dot{\vec{r}}^{\,2} - \frac{U}{\vec{r}^{\,2}} \\ &= -\frac{m}{2}\,\dot{\vec{r}}_0^{\,2} - \frac{U}{\vec{r}_0^{\,2}} \end{aligned}$$

by time invariance.

(b) $\dot{\vec{r}}^{\,*} = \dfrac{d\vec{r}^{\,*}}{dt^*} = \dfrac{(1+\varepsilon)\,d\vec{r}}{(1+\varepsilon)^2\,dt} = \dfrac{1}{1+\varepsilon}\,\dot{\vec{r}}$ and thus $\mathcal{L}^* = \dfrac{1}{(1+\varepsilon)^2}\,\mathcal{L}$, i.e.

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\mathcal{L}\left(t^*,x^*,\dot{x}^*\right)\cdot\frac{dt^*}{dt}\right) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathcal{L}^*\,\frac{dt^*}{dt} \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\frac{\mathcal{L}}{(1+\varepsilon)^2}\cdot(1+\varepsilon)^2 \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathcal{L} \\ &= 0 \end{aligned}$$

and the condition (10) of Noether's theorem holds.

(c) For the given transformations we have

$$\begin{aligned} x &\longmapsto x^* = (1+\varepsilon)x &&\Longrightarrow\quad \psi_x = x \\ y &\longmapsto y^* = (1+\varepsilon)y &&\Longrightarrow\quad \psi_y = y \\ z &\longmapsto z^* = (1+\varepsilon)z &&\Longrightarrow\quad \psi_z = z \\ t &\longmapsto t^* = (1+2\varepsilon)t &&\Longrightarrow\quad \varphi = 2t \end{aligned}$$

and so the Noether charge is given by

$$\begin{aligned}
Q(t,x,\dot{x}) &= \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\,\psi_i + \left(\mathcal{L} - \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\,\dot{x}_i \right) \varphi \\
&= \sum_{i=1}^{3} \frac{\partial}{\partial \dot{x}_i} \left(\frac{m}{2}\,\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}} \right) \psi_i + \\
&+ \left(\frac{m}{2}\,\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}} - \sum_{i=1}^{3} \frac{\partial}{\partial \dot{x}_i} \left(\frac{m}{2}\,\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}} \right) \dot{x}_i \right) \varphi \\
&= m(\dot{x}x + \dot{y}y + \dot{z}z) + \\
&+ \left(\frac{m}{2}\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}} - m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right) 2t \\
&= m\,\dot{\vec{r}}\,\vec{r} + \left(-\frac{m}{2}\dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}} \right) 2t \\
&= m\,\dot{\vec{r}}\,\vec{r} - (T+U)2t \\
&= m\,\dot{\vec{r}}\,\vec{r} - 2Et \\
&\overset{t=0}{=} m\,\dot{\vec{r}}_0\,\vec{r}_0
\end{aligned}$$

which shows that invariance under different transformations result in different conversation quantities.

4. Given the Heisenberg algebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \right\} = \langle X, Y, Z \,:\, [X,Y] = Z \rangle$$

and

$$\mathfrak{A}(\mathcal{H}) = \{\, \alpha \,:\, \mathcal{H} \longrightarrow \mathcal{H} \,:\, [\alpha(X), Y] = [\alpha(Y), X] \,\forall\, X, Y \in \mathcal{H} \,\}$$

Since $\mathfrak{A}(\mathcal{H})$ is a Lie algebra and

$$[X, \alpha] = [\operatorname{ad}(X), \alpha] = \alpha(X) \circ \alpha - \alpha \circ \mathrm{ad(X)}$$

a Lie multiplication, we can define

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H} \\
\mathcal{H}_{n+1} &:= \mathcal{H}_n \ltimes \mathfrak{A}(\mathcal{H}_{\mathfrak{n}})
\end{aligned}$$

and get a series of subalgebras

$$\mathcal{H}_0 \leq \mathcal{H}_1 \leq \mathcal{H}_2 \leq \ldots$$

Show that

(a) $\mathfrak{sl}(2) < \mathcal{H}_n$ is a proper subalgebra for all $n \geq 1$

(b) $\dim \mathcal{H}_n \geq 3 \cdot (2^{n+1} - 1)$ for all $n \geq 0$, i.e. the series is infinite and doesn't get stationary

As a counterexample, if we started with $\mathcal{H} = \mathfrak{su}(2)$ or $\mathfrak{su}(3)$ we would get $\mathcal{H}_n = \mathcal{H}_0$ and we were stationary right from the start, which can easily be seen by solving the corresponding system of linear equations.

Reason: Linear algebra.

Solution: To show that $\mathfrak{sl}(2)$ is a subalgebra of all $\mathcal{H}_n$ it is sufficient to show that

$$\alpha = \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{A}(\mathcal{H})$$

which is true, because

$$\begin{aligned} &[\alpha(xX + yY + zZ)\ ,\ x'X + y'Y + z'Z] \\ &= [(ax + by)X + (cx - ay)Y\ ,\ x'X + y'Y + z'Z] \\ &= (axy' + ayx' + byy' - cxx')Z \\ &= [\alpha(x'X + y'Y + z'Z)\ ,\ xX + yY + zZ] \end{aligned}$$

In general we have

$$\mathfrak{A}(\mathcal{H}_0) = \left\{ \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ x & y & z \end{bmatrix} \right\}$$

so it is a proper subalgebra for $\mathfrak{A}(\mathcal{H}_0)$ and for $n > 0$ by the following.

Next we show, that Z is a central element in all $\mathcal{H}_n$. This is clear for $n = 0$, so we assume that $Z \in \mathfrak{Z}(\mathcal{H}_n)$ is in the center of $\mathcal{H}_n$. Then we have for all $\alpha \in \mathfrak{A}(\mathcal{H}_n)$ and all $H \in \mathcal{H}_n$

$$[Z, \alpha](H) = [Z, \alpha(H)] - \alpha([Z, H]) = 0$$

and $Z \in \mathfrak{Z}(\mathcal{H}_n \ltimes \mathfrak{A}(\mathcal{H}_n)) = \mathfrak{Z}(\mathcal{H}_{n+1})$. Let $\{ h_1, \ldots, h_m \}$ be a basis for $\mathcal{H}_n\ ,\ n \geq 0$. Then

$$\alpha_i(a_1 h_1 + \ldots + a_m h_m) := a_i Z$$

define m linear independent transformations in $\mathfrak{A}(\mathcal{H}_n)$. For $m = 0$ we have $\dim \mathcal{H}_0 = \dim \mathcal{H} = 3 = 3 \cdot (2^1 - 1)$ and by induction

$$\begin{aligned}\dim \mathcal{H}_n &= \dim \mathcal{H}_{n-1} + \dim \mathfrak{A}(\mathcal{H}_{n-1}) \\ &\geq \dim \mathcal{H}_{n-1} + \dim \mathcal{H}_{n-1} + \dim \mathfrak{sl}(2) \\ &\geq 2 \cdot 3 \cdot (2^n - 1) + 3 \\ &= 3 \cdot (2^{n+1} - 1)\end{aligned}$$

because we have a copy of $\mathfrak{sl}(2)$ and a projection of every basis element on the central element Z.

5. A covering space $\tilde{X}$ of X is a topological space together with a continuous surjective map $p : \tilde{X} \longrightarrow X$, such that for every $x \in X$ there is an open neighborhood $U \subseteq X$ of x, such that $p^{-1}(U) \subseteq \tilde{X}$ is a union of pairwise disjoint open sets V_ι each of which is homeomorphically mapped onto U by p. A deck transformation with respect to p is a homeomorphism $h : \tilde{X} \longrightarrow \tilde{X}$ with $p \circ h = p$. Let $\mathcal{D}(p)$ be the set of all deck transformations with respect to p.

   (a) Show that $\mathcal{D}(p)$ is a group.

   (b) If $\tilde{X}$ is a connected Hausdorff space and $h \in \mathcal{D}(p)$ with $h(\tilde{x}) = \tilde{x}$ for some point $\tilde{x} \in \tilde{X}$. then $h = \operatorname{id}_{\tilde{X}}$.

   Reason: Coverings in set topology.

   Solution: (a) The homeomorphisms $\mathcal{H}(\tilde{X})$ build a group by successively applying the transformations. To an element $h : \tilde{X} \longrightarrow \tilde{X}$ we have the inverse $h^{-1} : \tilde{X} \longrightarrow \tilde{X}$ and the neutral element $\operatorname{id}_{\tilde{X}}$. Now the Deck transformations $\mathcal{D}(p) \leq \mathcal{H}(\tilde{X})$ is a subgroup, because

   - With $\operatorname{id}_{\tilde{X}} \in \mathcal{H}(\tilde{X})$ and $p(\operatorname{id}_{\tilde{X}}(\tilde{x})) = p(\tilde{x})$, we have $\operatorname{id}_{\tilde{X}} \in \mathcal{D}(p)$.
   - For $h, h' \in \mathcal{D}(p)$ we have $h \circ h' \in \tilde{\mathcal{X}}$ and

     $$p \circ (h^{-1} \circ h') = (p \circ h) \circ (h^{-1} \circ h') = p \circ (h \circ h^{-1}) \circ h' = p \circ h' = p$$

     and $D(p)$ is closed under inversion and multiplication.

   (b) Let $A := \{ \tilde{x} \in \tilde{X} : h(\tilde{x}) = \tilde{x} \} \neq \emptyset$. We have to show, that $A = \tilde{X}$ or equivalently due to connectedness, that A is open as well as closed.

   - If $h, h' : X \to Y$ are continuous functions and Y is a Hausdorff space, then $\{ x \in X : h(x) = h'(x) \}$ is closed. Since we can write $A = \{ \tilde{x} \in \tilde{X} : h(\tilde{x}) = \operatorname{id}_{\tilde{X}}(\tilde{x}) \}$ it is a closed set.

- Let $\tilde{x}_0 \in A$ and $x_0 := p(\tilde{x}_0)$. As p is a covering, there is an open neighborhood $x_0 \in U \subseteq X$ with equally many points in all fibers $p^{-1}(x)$, $x \in U$. We thus have $p^{-1}(U) = \coprod_{\iota \in I} V_\iota$ for a suited family $(V_\iota)_{\iota \in I}$ of open subsets of $\tilde{X}$, and the restrictions $p_\iota := p|_{V_\iota} : V_\iota \longrightarrow U$ are all homeomorphisms. Let $\iota_0 \in I$ be the index with $\tilde{x}_0 \in V_{\iota_0}$. Then $V := h^{-1}(V_{\iota_0}) \cap V_{\iota_0}$ is an open neighborhood of $\tilde{x}_0$ in $\tilde{X}$ since h is continuous. Since $\tilde{x}_0 \in A$, we have $\tilde{x}_0 = h(\tilde{x}_0) \in V_{\iota_0}$ and thus $\tilde{x}_0 \in h^{-1}(V_{\iota_0}) \cap V_{\iota_0} = V$. If $V \subseteq A$, then A is a neighborhood in $\tilde{X}$ of each of its points and therewith open.
- $V \subseteq A$.
  By definition of V we have $V \subseteq V_{\iota_0}$ and $h(V) \subseteq V_{\iota_0}$ and we can restrict h to a function $h_{\iota_0} : V \longrightarrow V_{\iota_0}$. From $p \circ h = p$ and $V \subseteq V_{\iota_0}$ we get for all $\tilde{x} \in V$

  $$p_{\iota_0}(\tilde{x}) = p(\tilde{x}) = p(h(\tilde{x})) = p(h_{\iota_0}(\tilde{x})) = p_{\iota_0}(h_{\iota_0}(\tilde{x}))$$

  Since $p_{\iota_0} : V_{\iota_0} \longrightarrow U$ is a homeomorphism and in particular injective, we have $h(\tilde{x}) = h_{\iota_0}(\tilde{x}) = \tilde{x}$ for all $\tilde{x} \in V$ and thus $V \subseteq A$.

8 June-B 2018

1. The general solution to $y^{(4)}(x) + 4y(x) = 0$ is given by

$$y(x) = \alpha e^{-x}\cos(x) + \beta e^{-x}\sin(x) + \gamma e^{x}\sin(x) + \delta e^{x}\cos(x)$$

   (a) How do the initial conditions at $x = 0$ have to be chosen in order to get $y(x) = e^{-x}\cos x$ as unique solution?

   (b) Which function do we get for the initial conditions $y'(0) = 1\ ,\ y''(0) = 0\ ,\ y'''(0) = 0\ ,\ y^{(4)}(0) = 0\,?$

Reason: Initial conditions are essential.

Solution:

$$\begin{aligned}
y'(x) &= -\alpha e^{-x}(\sin(x) + \cos(x)) + \beta e^{-x}(\cos(x) - sin(x)) + \\
&\quad + \gamma e^{x}(\sin(x) + \cos(x)) + \delta e^{x}(\cos(x) - \sin(x)) \\
y''(x) &= 2\alpha e^{-x}\sin(x) - 2\beta e^{-x}\cos(x) + 2\gamma e^{x}\cos(x) - 2\delta e^{x}\sin(x) \\
y'''(x) &= -2\alpha e^{-x}(\sin(x) - \cos(x)) + 2\beta e^{-x}(\sin(x) + \cos(x)) + \\
&\quad - 2\gamma e^{x}(\sin(x) - \cos(x)) - 2\delta e^{x}(\sin(x) + \cos(x)) \\
y^{(4)}(x) &= -4\alpha e^{-x}\cos x - 4\beta e^{-x}\sin x - 4\gamma e^{x}\sin x - 4\delta e^{x}\cos x
\end{aligned}$$

As we are interested in initial conditions at $x = 0$ all terms $e^{\pm x} = 1\ ,\ \sin(x) = 0\ ,\ \cos(x) = 1$ and we get the following linear equation system

$$\begin{bmatrix} y'(0) \\ y''(0) \\ y'''(0) \\ y^{(4)}(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 0 \\ 2 & 2 & 2 & -2 \\ -4 & 0 & 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

that means in order to get $y(x) = 1 \cdot e^{-x}\cos(x)$ we have to chose $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 2$, $y^{(4)}(0) = -4$ and with

$$\begin{aligned}
\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix} &= \begin{bmatrix}-1 & 1 & 1 & 1\\ 0 & -2 & 2 & 0\\ 2 & 2 & 2 & -2\\ -4 & 0 & 0 & -4\end{bmatrix}^{-1} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\
&= \frac{1}{8}\cdot \begin{bmatrix}-2 & 0 & 1 & -1\\ 2 & -2 & 1 & 0\\ 2 & 2 & 1 & 0\\ 2 & 0 & -1 & -1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\
&= \frac{1}{4}\cdot \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}
\end{aligned}$$

we get for the given initial conditions

$$y(x) = -\frac{1}{4}e^{-x}\cos(x) + \frac{1}{4}e^{-x}\sin(x) + \frac{1}{4}e^{x}\sin(x) + \frac{1}{4}e^{x}\cos(x)$$

2. Calculate

$$\int_{\pi^{-1}}^{\pi} \frac{1}{x}\sin^2\left(-x-\frac{1}{x}\right)\log x\,dx$$

Reason: Symmetries can be multiplicative, too.

Solution: We substitute $u = x^{-1}$, $du = -x^{-2}\,dx = -u^2\,dx$ and get

$$\begin{aligned}
\mathcal{I} &= \int_{\pi^{-1}}^{\pi} \sin^2\left(-x-\frac{1}{x}\right)\cdot\frac{\log x}{x}\,dx\\
&= \int_{\pi}^{\pi^{-1}} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{-\log u}{u^{-1}}\cdot(-u^{-2})\,du\\
&= \int_{\pi}^{\pi^{-1}} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{\log u}{u}\,du\\
&= -\int_{\pi^{-1}}^{\pi} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{\log u}{u}\,du\\
&= -\mathcal{I}
\end{aligned}$$

3. Decryption:

   Reason: Fun (easy - diligence).

   (a) "ZC ULX QFFY L TBXCFSB FMFS XZYVF ZC RLX AZXVDMFSFA TDSF CULY KZKCB BFLSX LPD, LYA LOO PDDA CUFDS-FCZVLO IUBXZVZXCX IHC CUZX YHTQFS HI DY CUFZS RLOO LYA RDSSB LQDHC ZC"

   Solution: It is an easy Caesar code ($x \longmapsto 5x+7$) and means

   "It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it" (R. Feynman)

   (b) "CO PXLIX BPX Y FPRDMCPO MP KPDOM, Y BDRR LIX-CAYMCPO PX VXPPB SDFM ZI WCAIO. YOFJIXF JCMU OP VXPPB JCRR ZI CWOPXIL. CM CF BCOI MP DFI OPOMX-CACYR XIFDRMF JCMUPDM VXPPB YF RPOW YF EPD KCMI MUIS YOL YF RPOW YF CM CF KPSSPO QOPJRILWI MP YRR SYMUISYMCKCYOF. JUIMUIX MUI RYMMIX CF FYMCFBCIL JCRR ZI LIKCLIL PO Y KYFIZEKYFI ZYFCF. CB EPD UYAI FIIO MUI VXPZRIS ZIBPXI YOL XISISZIX MUI FPRDMCPO, EPD KYOOPM VYXMCKCVYMI CO MUI FPRDMCPO MP MUYM VXPZRIS"

   Solution: It's a randomly chosen mapping of the alphabet onto itself.

A	1	25	Y
B	2	26	Z
C	3	11	K
D	4	12	L
E	5	9	I
F	6	2	B
G	7	23	W
H	8	21	U
I	9	3	C
J	10	20	T
K	11	17	Q
L	12	18	R
M	13	19	S
N	14	15	O
O	15	16	P
P	16	22	V
Q	17	8	H
R	18	24	X
S	19	6	F
T	20	13	M
U	21	4	D
V	22	1	A
W	23	10	J
X	24	14	N
Y	25	5	E
Z	26	7	G

"In order for a solution to count, a full derivation or proof must be given. Answers with no proof will be ignored. It is fine to use nontrivial results without proof as long as you cite them and as long as it is "common knowledge to all mathematicians". Whether the latter is satisfied will be decided on a case-by-case basis. If you have seen the problem before and remember the solution, you cannot participate in the solution to that problem."

4. Farmer Joe bought the blue area for \$10,000, the green for \$20,000 and the yellow for \$30,000. Assuming prices are proportional to the areas, what's the price for his entire field?

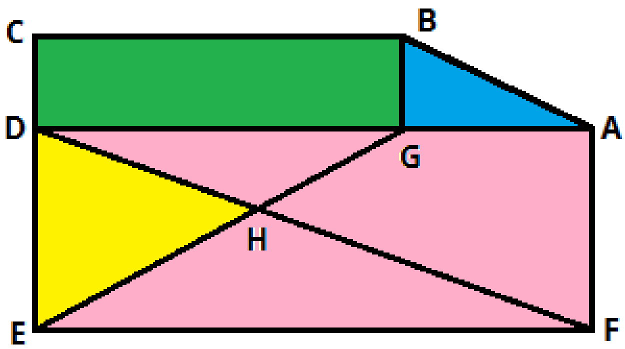


Figure 3:
Farmer Joe's Field

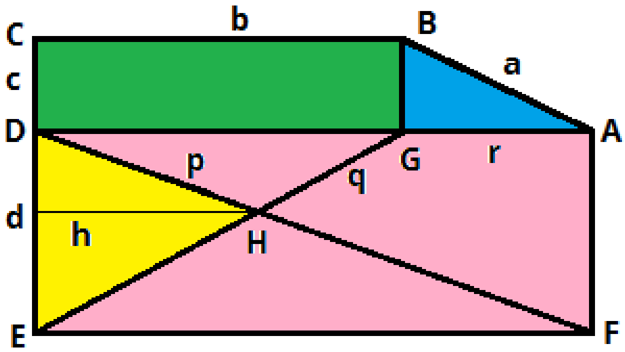


Figure 4:
Farmer Joe's Field

Reason: Geometry. Find the clue.

Solution: We will use the following notation: Thus we know that $bc = 2 \cdot \frac{1}{2} \cdot cr$, i.e. $r = b$. For the intersection of p and q we get the equation of straight lines by

$$\frac{d}{b} \cdot h = -\frac{d}{b+r} \cdot h + d$$

and therefore $h = \frac{2}{3}b$. Since the yellow area is three times the blue area, we have

$$\frac{1}{2}dh = \frac{1}{2} \cdot \frac{2}{3}db = 3 \cdot \frac{1}{2}bc$$

$d = \frac{9}{2}c$, and thus for the total area

$$A_{ABCEF} = (c+d) \cdot 2b - \frac{1}{2}bc = \frac{21}{2}bc = \frac{21}{2} \cdot \$20,000 = \$210,000$$

5. Inequalities.

   Reason: Useful.

   (a) Prove $(e+x)^{e-x} > (e-x)^{e+x}$ for $0 < x < e$.

   Solution: We define

$$f\,:\,]0,e[\longrightarrow \mathbb{R}\,,\; f(x) := (e-x)\log(e+x) - (e+x)\log(e-x)$$

   and observe $\lim_{x\searrow 0} f(x) = 0$. Then

$$\begin{aligned} f\,'(x) &= \frac{e-x}{e+x} + \frac{e+x}{e-x} - (\log(e+x) + \log(e-x)) \\ &= 2\cdot \underbrace{\frac{e^2+x^2}{e^2-x^2}}_{>1} - \underbrace{\log(e^2-x^2)}_{<2} \\ &> 0 \end{aligned}$$

   and $f(x) > 0$, resp. $(e-x)\log(e+x) > (e+x)\log(e-x)$ resp. $(e+x)^{e-x} > (e-x)^{e+x}$

   (b) Show that for $0 < b < a$ we have

$$\frac{1}{a} < \frac{2}{a+b} < \frac{\log(a)-\log(b)}{a-b} < \frac{1}{\sqrt{ab}} < \frac{1}{b}$$

   Solution: For $x = \dfrac{a}{b} > 1$ we have

$$\begin{aligned} \log(x^2) &= 2\log(x) \\ &= 2\int_1^x \frac{1}{t}\,dt \\ &< \int_1^x \left(1+\frac{1}{t^2}\right)\,dt \\ &= x - \frac{1}{x} \end{aligned}$$

   or

$$\log(x) < \sqrt{x} - \frac{1}{\sqrt{x}}$$

   With $\log(x) = \sum_{k=0}^{\infty} \frac{2}{2k+1}\left(\dfrac{x-1}{x+1}\right)^{2k+1} > 2\cdot\dfrac{x-1}{x+1}$ we get

$$2\frac{\frac{a}{b}-1}{\frac{a}{b}+1} = 2\frac{a-b}{a+b} < \log(\frac{a}{b}) = \log(a) - \log(b) < \sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} = \frac{a-b}{\sqrt{ab}}$$

(c) Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two monotone integrable functions, either both increasing or both decreasing. Show that

$$\int_a^b f(x)g(x)\,dx \geq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

Solution: We have $(f(x) - f(y))\,(g(x) - g(y)) > 0$ and therefore

$$\int_a^b f(x)g(x)\,dx + \int_a^b f(y)g(y)\,dy \geq$$
$$\int_a^b f(x)\,dx \int_a^b g(y)\,dy + \int_a^b f(y)\,dy \int_a^b g(x)\,dx$$

or

$$2\int_a^b f(x)g(x)\,dx \geq 2\int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

9 June-I 2018

1. Determine with analytical methods, i.e. with a calculator only, the wavelengths of all local maximal radiation intensities of a black body of temperature T given the following function of radiation intensity up to three digits.

$$J(\lambda) = \frac{c^2 h}{\lambda^5 \cdot \left(\exp\left(\frac{ch}{\lambda\kappa T}\right) - 1\right)}$$

Reason: Physics.

Solution: With $x = \frac{\kappa T}{ch}\lambda$ we get for $x > 0$

$$F(x) = x^{-5}\left(e^{\frac{1}{x}} - 1\right)^{-1} = \frac{c^3h^4}{\kappa^5T^5}\,J(x)$$
$$F'(x) = \left(e^{\frac{1}{x}} - 5x\left(e^{\frac{1}{x}} - 1\right)\right) \cdot x^{-7} \cdot \left(e^{\frac{1}{x}} - 1\right)^{-2}$$

and $F'(x) = 0$ if and only if $e^{\frac{1}{x}} - 5x\left(e^{\frac{1}{x}} - 1\right) = 0$ or

$$f(t) := 5(1 - e^{-t}) = t \text{ with } t = x^{-1}$$

Because of $f'(t) > 1$ on $[0, \log 5]$ the function $f(t) - t$ is strictly monotone increasing there and $f(t) > t$ since $f(0) = 0$. For $t > \log 5$ we get that $f(t)-t$ is strictly monotone decreasing and thus has at most one zero. As $f(4)-4>0$ and $f(5)-5<0$ there is exactly one zero t^* in $[4,5]$ by the intermediate value theorem. Now

$$q := \sup\{\, |f'(t)| \,:\, t\in[4,5]\,\} = f'(4) = 5e^{-4} = 0.09157 < 1$$

and by the fixed-point theorem and a sequence $t_{n+1} := f(t_n)\ ,\ t_1 = 5$ we have

$$|t^* - t_n| < \frac{q}{q-1}\ |t_n - t_{n-1}| < 0.1008\ |t_n - t_{n-1}|$$

and thus $t^* = 4.965114 \pm 10^{-6}$ or $x^* = 0.2014052 \pm 10^{-7}$. Since $\lim_{x\searrow 0} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 0$ it is the only maximum and

$$\begin{aligned} \lambda^* &= \frac{ch}{\kappa T}\ x^* \\ &= 0.2014\ \frac{ch}{\kappa T} \\ &= 0.2014\ \cdot\ 1.4388\ \text{cm}\cdot T^{-1} \\ &= 0,28977\ \text{cm}\cdot T^{-1} \end{aligned}$$

2. Consider $\mathfrak{su}(3) = \operatorname{span}\{\, T_3, Y, T_\pm, U_\pm, V_\pm \,\}$ given by the basis elements

$$\begin{aligned} &T_3 = \frac{1}{2}\lambda_3\ ,\ Y = \frac{1}{\sqrt{3}}\lambda_8\ , \\ &T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda - 2)\ ,\ U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)\ ,\ V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5) \end{aligned}$$

(cp. https://www.physicsforums.com/insights/representations-precision-important) where the λ_i are the Gell-Mann matrices and its maximal solvable Borel-subalgebra

$$\mathfrak{B} := \langle T_3, Y, T_+, U_+, V_+ \rangle$$

Now $\mathfrak{A}(\mathfrak{B}) = \{\, \alpha : \mathfrak{g} \to \mathfrak{g} \,:\, [X,\alpha(Y)] = [Y,\alpha(X)]\ \forall\, X,Y \in \mathfrak{B} \,\}$ is the one-dimensional Lie algebra spanned by $\operatorname{ad}(V_+)$ because $\mathbb{C}V_+$ is a one-dimensional ideal in $\mathfrak{B}$ (Proof?). Then $\mathfrak{g} := \mathfrak{B} \ltimes \mathfrak{A}(\mathfrak{B})$ is again a Lie algebra by the multiplication $[X,\alpha] = [\operatorname{ad} X, \alpha]$ for all $X \in \mathfrak{B}\ ,\ \alpha \in \mathfrak{A}(\mathfrak{B})$. (For a proof see problem 9 in https://www.physicsforums.com/threads/intermediate-math-challenge-may-2018.946386/)

(a) Determine the center of $\mathfrak{g}$, and whether it is semisimple, solvable, nilpotent or neither.

(b) Show that $(X,Y) \mapsto \alpha([X,Y])$ defines another Lie algebra structure on $\mathfrak{B}$, which one?

(c) Show that $\mathfrak{A}(\mathfrak{g})$ is at least two-dimensional.

Reason: One-dimensional representation of $\mathfrak{B}$.

Solution: We have the following multiplication table

$$[T_3,Y]=[T_+,Y]=[T_+,V_+]=[U_+,V_+]=0$$
$$[T_3,T_+]=T_+\ ,\ [T_3,U_+]=-\frac{1}{2}U_+\ ,\ [T_3,V_+]=\frac{1}{2}V_+$$
$$[U_+,T_+]=-V_+\ ,\ [Y,U_+]=U_+\ ,\ [Y,V_+]=V_+$$

With $\mathfrak{A}(\mathfrak{B})=\mathbb{C}\cdot\alpha\ ,\ \alpha(Z)=\operatorname{ad}V_+(Z)=[V_+,Z]$ we get

$$[X,\alpha]=X.\alpha=[\operatorname{ad}X,\operatorname{ad}V_+]=\operatorname{ad}[X,V_+]\sim\operatorname{ad}V_+\sim\alpha$$

and $\operatorname{span}\{T_+,U_+,V_+\}\subseteq\ker\alpha=\ker\operatorname{ad}V_+=\mathfrak{C}_{\mathfrak{B}}(V_+)$, so

$$\begin{aligned}
\mathfrak{g}^{(0)}&=\mathfrak{g}=\mathfrak{B}\oplus\mathfrak{A}(\mathfrak{B})\\
\mathfrak{g}^{(1)}&=[\mathfrak{g},\mathfrak{g}]=[\mathfrak{B},\mathfrak{B}]\oplus\mathfrak{A}(\mathfrak{B})=\langle T_+,U_+,V_+\rangle\oplus\mathfrak{A}(\mathfrak{B})\\
\mathfrak{g}^{(2)}&=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]=\mathbb{C}V_+\oplus\{0\}\\
\mathfrak{g}^{(3)}&=[\mathfrak{g}^{(2)},\mathfrak{g}^{(2)}]=\{0\}
\end{aligned}$$

Therefore $\mathfrak{g}=\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B})$ is solvable, and not semisimple. If we take a central element $Z=aT_3+bY+cT_++dU_++eV_++f\alpha\in\mathfrak{Z}(\mathfrak{g})$ and solve successively

$$[Z,U_+]=0\to[Z,V_+]=0\to[Z,Y]=0$$

then we get all coefficients have to be zero, i.e. $\mathfrak{Z}(\mathfrak{g})=\{0\}$ and $\mathfrak{g}$ cannot be nilpotent. It also shows, that $\alpha([X,Y])=0$ is an Abelian structure on $\mathfrak{B}$.

For a one-dimensional ideal $\mathfrak{I}=\langle V_0\rangle$ of any Lie algebra $\mathfrak{h}$ we have $[X,V_0]=\mu(X)V_0$ for all $X\in\mathfrak{h}$ and some linear form $\mu\in\mathfrak{h}^*$. With $\alpha(X):=\operatorname{ad}(V_0)(X)=-\mu(X)V_0$ we always get a non-trivial antisymmetric transformation of $\mathfrak{h}$. Therefore $\beta_1(B+b\alpha):=-\mu(X)V_+$ defines a non-trivial antisymmetric transformation of $\mathfrak{g}=\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B})$, since $\mathfrak{I}=\mathbb{C}\cdot V_+\lhd\mathfrak{g}$ is a one-dimensional ideal. However, $\mathbb{C}\cdot\alpha=\mathfrak{A}(\mathfrak{B})$ is also a one-dimensional ideal of $\mathfrak{g}$, so $\beta_2(B+b\alpha):=\mu(X)\alpha$ is antisymmetric, too, and linear independent of β_1. Thus

$$\dim\mathfrak{A}(\mathfrak{g})=\mathfrak{A}(\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B}))\geq 2$$

3. Consider the Hilbert space $\mathcal{H} = L_2([a,b])$ of Lebesgue square integrable functions on $[a,b]$, i.e.

$$\langle \psi, \chi \rangle = \int_a^b \psi(x)\chi(x)\,dx$$

The functions $\{\, \psi_n := x^n \,:\, n \in \mathbb{N}_0 \,\}$ build a system of linear independent functions which can be used to find an orthonormal basis by the Gram-Schmidt procedure.

Show that the Legendre polynomials

$$p_n(x) := \frac{1}{(b-a)^n\, n!}\sqrt{\frac{2n+1}{b-a}}\,\frac{d^n}{dx^n}[(x-a)(x-b)]^n \;,\; n \in \mathbb{N}_0$$

build an orthnormal system.

Reason: L_2 spaces. Educational.

Solution: We first show $\langle p_n, p_m \rangle = 0$ for $n < m$. As $\deg p_n(x) \leq n$ it is sufficient to show by successively integration by parts for $n < m$

$$\begin{aligned}
\int_a^b x^n p_m(x)\,dx &= C_m \int_a^b x^n \,\frac{d^m}{dx^m}[(x-a)(x-b)]^m\,dx \\
&= C_m \left[x^n \,\frac{d^{m-1}}{dx^{m-1}}[(x-a)(x-b)]^m \right]_a^b \\
&- \;C_m \int_a^b \frac{d}{dx}\,x^n \,\frac{d^{m-1}}{dx^{m-1}}\,[(x-a)(x-b)]^m\,dx \\
&= 0 - C_m \int_a^b \frac{d}{dx}\,x^n \,\frac{d^{m-1}}{dx^{m-1}}\,[(x-a)(x-b)]^m\,dx \\
&\;\vdots \\
&= C_m\,(-1)^{n+1} \int_a^b \frac{d^{n+1}}{dx^{n+1}}\,x^n \,\frac{d^{m-n-1}}{dx^{m-n-1}}\,[(x-a)(x-b)]^m\,dx \\
&= 0
\end{aligned}$$

Now we have to show that $||p_n|| = 1$, where we use again integration by parts and the fact that all boundary conditions vanish as long as terms $(x-a)(x-b)$ occur. Equivalently

$$
\begin{aligned}
&\left\|\frac{d^n}{dx^n}[(x-a)(x-b)]^n\right\|^2\\
&=\int_a^b\left[\frac{d^n}{dx^n}[(x-a)(x-b)]^n\right]^2\,dx\\
&=-\int_a^b\left[\frac{d^{n-1}}{dx^{n-1}}[(x-a)(x-b)]^n\right]\cdot\left[\frac{d^{n+1}}{dx^{n+1}}[(x-a)(x-b)]^n\right]\,dx\\
&\;\;\vdots\\
&=(-1)^n\int_a^b(x-a)^n(x-b)^n\frac{d^{2n}}{dx^{2n}}[(x-a)(x-b)]^n\\
&=(2n)!\,(-1)^n\int_a^b(x-a)^n(x-b)^n\,dx\\
&=(2n)!\,(-1)^n\,(-1)^1\,\frac{n}{n+1}\int_a^b(x-a)^{n+1}(x-b)^{n-1}\,dx\\
&\;\;\vdots\\
&=(2n)!\,(-1)^n\,(-1)^n\,\frac{n\cdot(n-1)\cdot\ldots\cdot 1}{(n+1)\cdot(n+2)\cdot\ldots\cdot(2n)}\int_a^b(x-a)^{2n}\,dx\\
&=(n!)^2\int_a^b(x-a)^{2n}\,dx\\
&=(n!)^2\frac{1}{2n+1}\left[(x-a)^{2n+1}\right]_a^b\\
&=\frac{(n!)^2}{2n+1}(b-a)^{2n+1}
\end{aligned}
$$

4. Rings.

   (a) Give an example of an integral domain (no field), which has common divisors, but doesn't have greatest common divisors.

   (b) Show that there are infinitely many units (invertible elements) in $\mathbb{Z}[\sqrt{3}]$.

   (c) Determine the units of $\{\,\frac{1}{2}a+\frac{1}{2}b\sqrt{-3}\mid a+b \text{ even }\}$.

   (d) The ring R of integers in $\mathbb{Q}(\sqrt{-19})$ is the ring of all elements, which are roots of monic polynomials with integer coefficients. Show that R is built by all elements of the form $\frac{1}{2}a+\frac{1}{2}b\sqrt{-19}$ where $a,b\in\mathbb{Z}$ and both are either even or both are odd.

   Reason: World outside of $\mathbb{Z}$ and $\mathbb{C}$.

Solution:

(a) $R := \mathbb{Z}[\sqrt{-5}]$.

$$1, 3, 2+\sqrt{-5}, 2-\sqrt{-5} \mid 9 = 3\cdot 3 = (2+\sqrt{-5})\cdot(2-\sqrt{-5})$$

and there is no greatest common divisor in this set of divisors.

(b) All numbers $\pm(2\pm\sqrt{3})^n$, $n \in \mathbb{N}_0$ are units in $\mathbb{Z}\sqrt{3}$.

(c) From $|\frac{1}{2}a + \frac{1}{2}b\sqrt{-3}|^2 = 1$ for units, we get $4 = a^2 + 3b^2$, i.e. $|a| = 2, b = 0$ or $|a| = |b| = 1$, which are the six elements $\left[\frac{1}{2}(1+\sqrt{-3})\right]^n$, $n = 0, \ldots, 5$.

(d) For $r = \frac{1}{2}(a+b\sqrt{-19}) \in R$ we have $\frac{1}{2}(a+b\sqrt{-19})\cdot\frac{1}{2}(a-b\sqrt{-19}) = \frac{1}{4}(a^2+19b^2) \in \mathbb{Z}$, because $a+b \equiv 0(2)$. So

$$(x-\frac{1}{2}(a+b\sqrt{-19}))(x-\frac{1}{2}(a-b\sqrt{-19})) = x^2-ax+\frac{1}{4}(a^2+19b^2) \in \mathbb{Z}[x]$$

If we have $r \in \mathbb{Q}(\sqrt{-19})$ an integer, then $r^2 + ar + b = 0$ for $a, b \in \mathbb{Z}$, i.e. $2r = -a \pm \sqrt{a^2-4b}$. As $r \in \mathbb{Q}(\sqrt{-19})$ we have $\mathbb{Z} \ni a^2 - 4b = (\alpha + \beta\sqrt{-19})^2 = \alpha^2 - 19\beta^2 + 2\alpha\beta\sqrt{-19}$ and thus $\alpha\beta = 0$.

Case 1: $\beta = 0$. Then $b = \frac{a-\alpha}{2}\cdot\frac{a+\alpha}{2} \in \mathbb{Z}$ and $r_{1,2} = -\frac{1}{2}(a\pm\alpha)$ and $a \pm \alpha \equiv 0(2)$ which means $r_{1,2} \in R$.

Case 2: $\alpha = 0$. Then $b = \frac{1}{4}(a^2 + 19\beta^2) \in \mathbb{Z}$ and $r_{1,2} = -\frac{1}{2}(a \pm \beta\sqrt{-19})$ and we have to show that $-a \pm \beta \equiv 0(2)$. Let us assume this is not the case and $\beta^2 = (2k+a+1)^2$. Then

$$4 \,|\, (a^2 + 19\beta^2) = 20a^2 + 76\cdot(n^2+an+n) + 38a + 19 \not\equiv 0(4)$$

which is a contradiction. Therefore we have again $r_{1,2} \in R$.

10 May-B 2018

1. Finite Field $\mathbb{F}_8$.

   Reason: Do not always assume $\operatorname{char}\mathbb{F} = 0$.

   (a) Find a minimal polynomial to determine the factor ring which is isomorphic to $\mathbb{F}_8$.

   Solution: $\mathbb{F}_8$ is three dimensional as $\mathbb{F}_2$ vector space. Therefore the minimal polynomial is of degree three and $m(x;\mathbb{F}_2) = x^3+x+1$ is irreducible because neither element of $\mathbb{F}_2 = \{0,1\}$ is a zero. We then have $\mathbb{F}_8 \cong \mathbb{F}_2[x]/(x^3+x+1)$.

   (b) From there determine a basis of $\mathbb{F}_8$ over $\mathbb{F}_2$ and write down its multiplication and addition laws.

   Solution: Let $\xi^3+\xi+1 = 0$. Then

   $$m(x;\mathbb{F}_2) = (x+\xi)(x+\xi^2)(x+\xi+\xi^2)$$

   and $\{1,\xi,\xi^2\}$ is a $\mathbb{F}_2$ basis of $\mathbb{F}_8$. The elements are thus

   $$\{0,1,\xi,\xi+1=\xi^3,\xi^2,\xi^2+1=\xi^6,\xi^2+\xi=\xi^4,\xi^2+\xi+1=\xi^5\}$$

   which defines the multiplicative group generated by ξ as well as the addition table.

   (c) Why does the algebraic closure of a finite field have to be infinite?

   Solution: We get a field $\mathbb{F}_{p^n}$ for every natural number $n \in \mathbb{N}$ over the prime field $\mathbb{F}_p$ which is algebraic over $\mathbb{F}_p$ of dimension n. Since the algebraic closure has to contain all of them, it has to be infinite, although of (prime) characteristic $p \neq 0$.

2. Determine the open balls with radius 3 around $(2,1) \in \mathbb{R}^2$ w.r.t.

   Reason: Open neighborhoods can be very different.

   (a) the French Railway metric with Paris at the origin P and Reims at $R = (2,1)$.

Solution: The distance between two points A, B is defined by

$$d(A,B) = \begin{cases} ||A-B||_2 & \text{if } \overline{AP} = \overline{BP} \\ ||A||_2 - ||B||_2 & \text{in all other cases} \end{cases}$$

All distances between points have to include Paris, so we have $3-\sqrt{5}$ left if we have to travel to Paris first and the full 3 if we travel on $R \cdot \lambda$ outbound. This results in

$$B_3\,(R;d) = B_{3-\sqrt{5}}\,(P;||\,.\,||_2) \cup \left\{ R\cdot\lambda \,:\, 1-\frac{3}{\sqrt{5}} < \lambda < 1+\frac{3}{\sqrt{5}} \right\}$$

The open ball of radius three around Reims includes all the way to Luxembourg in one direction and to all points in the inner highway circle of Paris. The good news is, that we can't reach Saint-Quentin although it is nearer than Paris.

(b) the Manhattan metric

Solution: The distance between two points A, B is defined by

$$d(A,B) = \sum_i |a_i - b_i|$$

and thus we get an open rhombus with vertices $(-1,1)$, $(2,4)$ $(5,1)$ and $(2,-2)$.

(c) the maximum metric

Solution: The distance between two points A, B is defined by

$$d(A,B) = \max\{|a_i - b_i| \,:\, i = 1,2\}$$

and thus we get an open square with vertices $(-1,-2)$, $(-1,4)$ $(5,4)$ and $(5,-2)$.

3. Calculate the volume $\mu(A)$ of

$$A = \{(x,y,z) \in \mathbb{R}^3 \,:\, x,y,z \geq 0\,,\; x+y+z \leq \sqrt{2}\,,\; x^2+y^2 \leq 1\,\}$$

Reason: Practice volume integrals.

Solution: The volume is given by $\mu(A) = \int_A d\mu$. The first reduction of A by z gives us

$$A' = \{\{(x,y) \in \mathbb{R}^2 \,:\, x,y \geq 0\;,\; x^2+y^2 \leq 1\,\}\;;\; A_{(x,y)} = [0, \sqrt{2}-x-y]$$

and the second by y

$$A'' = [0,1]\;;\; A'_x = [0, \sqrt{1-x^2}]$$

We thus have

$$\begin{aligned}
\mu(A) &= \int_A d\mu \\
&= \int_{A'} \left(\int_0^{\sqrt{2}-x-y} dz \right) d\mu_{A'} \\
&= \int_{A''} \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{2}-x-y} dz \right) dy \right) d\mu_{A''} \\
&= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{2}-x-y} dz \right) dy \right) dx \\
&= \int_0^1 \left[(\sqrt{2}-x)\sqrt{1-x^2} - \frac{1}{2}(1-x^2) \right] dx \\
&= \sqrt{2}\int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 x\sqrt{1-x^2}\,dx - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1 x^2\,dx \\
&= \sqrt{2}\cdot\frac{\pi}{4} - \frac{1}{3} - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} \\
&= \frac{\pi}{2\sqrt{2}} - \frac{2}{3}
\end{aligned}$$

4. Solve $\int_\Gamma \omega$ with the curve $\Gamma = \gamma([0,1])$ given by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3\;,\; \gamma(t) = (t^2, 2t, 1) \text{ and } \omega = z^2dx + 2ydy + xzdz$$

and compute the exterior derivative $\nu = d\omega$. As such, the result is an exact 2−form. Is it also closed? Show this by calculation.

Reason: Differential forms.

Solution:

$$\begin{aligned}
\int_\Gamma \omega &= \int_{[0,1]} \gamma^*(\omega) \\
&= \int_{[0,1]} \omega(d\gamma) \\
&= \int_{[0,1]} (z^2dx + 2ydy + xzdz)d\gamma \\
&= \int_{[0,1]} (1^2\,dx + 2\cdot 2t\,dy + t^2\cdot 1\,dz)d\gamma \\
&= \int_{[0,1]} \left(1^2\frac{d(t^2)}{dt} + 2(2t)\frac{d(2t)}{dt} + t^2\cdot 1\frac{d(1)}{dt}\right)\,dt \\
&= \int_0^1 (2t + 8t + 0)\,dt \\
&= \int_0^1 10t\,dt \\
&= 5
\end{aligned}$$

$$\begin{aligned}
d\omega &= d(z^2dx + 2ydy + xzdz) \\
&= (\frac{\partial}{\partial x}z^2dx + \frac{\partial}{\partial y}z^2dy + \frac{\partial}{\partial z}z^2dz) \\
&\wedge dx + (\frac{\partial}{\partial x}2ydx + \frac{\partial}{\partial y}2ydy + \frac{\partial}{\partial z}2ydz) \\
&\wedge dy + (\frac{\partial}{\partial x}xzdx + \frac{\partial}{\partial y}xzdy + \frac{\partial}{\partial z}xzdz)?dz \\
&= 2zdz\wedge dx + 2dy\wedge dy + (zdx + xdz)\wedge dz \\
x &= 2zdz\wedge dx - zdz\wedge dx \\
&= zdz\wedge dx
\end{aligned}$$

$$\begin{aligned}
d\nu &= d(zdz\wedge dx) \\
&= d(-1dx\wedge zdz) \\
&= \left(\frac{\partial}{\partial x}(-1)\,dx + \frac{\partial}{\partial y}(0)\,dy + \frac{\partial}{\partial z}(z)\,dz\right)\wedge dx\wedge dz \\
&= dz\wedge dx\wedge dz \\
&= 0
\end{aligned}$$

11 May-I 2018

1. Solve $\mathcal{I} = \int_{-1}^{0} x \cdot \sqrt{x^2+x+1}\,dx$. Hint: Use $\cosh^2 x - \sinh^2 x = 1$.

   Reason: Practice integration.

   Solution: (134.f.E2.) We have $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$ and we set $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\sinh u$. Thus $dx = \frac{\sqrt{3}}{2}\cosh u\,du$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\cosh u$ with the hint given. All together we get

   $$\mathcal{I} = \int_{-1}^{0} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\sinh u\right)\left(\frac{\sqrt{3}}{2}\cosh u\right)\left(\frac{\sqrt{3}}{2}\cosh u\right) du$$

   Integration by parts and again using the hint results in

   $$\mathcal{I} = \frac{\sqrt{3}}{8}\cosh^3 u - \frac{3}{16}\sinh u\,\cosh u - \frac{3}{16}u$$

   Now re-substitution with $\sinh u = \frac{2}{\sqrt{3}}(x+\frac{1}{2})$, $\cosh u = \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$ and with the inverse function $\operatorname{arsinh} v = \log(v+\sqrt{v^2+1})$ we have

   $$u = \operatorname{arsinh}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) = \log\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right)$$

   and for our integral (modulo constant terms)

   $$\mathcal{I} = \frac{1}{3}\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{1}{4}(x+\frac{1}{2})\sqrt{x^2+x+1} - \frac{3}{16}\log(x+\frac{1}{2}+\sqrt{x^2+x+1})$$

   which from $x=-1$ to $x=0$ results in

   $$\mathcal{I} = -\frac{1}{4} - \frac{3}{16}\log 3 \approx -0.45598980\ldots \approx -0.456$$

2. Given the differential operators $D_n := x^n \cdot \dfrac{d}{dx}$ $(n \in \mathbb{Z})$ on smooth real valued functions $\mathcal{C}^\infty(\mathbb{R})$. Determine for which subsets $L \subseteq \mathbb{Z}$ the set $\{D_n \,|\, n \in L\}$ is a basis for a finite dimensional Lie algebra and which Lie algebra is it.

   Reason: Differential structures on the real line.

   Solution: For $f \in \mathcal{C}^\infty(\mathbb{R})$ we get

   $$D_n(D_m.f) = D_n(x^m f') = x^{n+m}f'' + mx^{n+m-1}f'$$

and thus

$$[D_n, D_m].f = (m-n)x^{n+m-1}f\,' = (m-n)D_{n+m-1}.f$$

In order to get a finite dimensional Lie algebra, the set of all $L = \{n, m, n+m-1\}$ must be finite.

- The easiest examples are $n = m$ with $[D_n, D_n] = 0$, the one dimensional Abelian Lie algebra. It's the only possibility with commuting operators.
- We also observe, that $[D_1, D_n] = (n-1)D_n$. So for $n \neq 1$ this yields the two dimensional non Abelian Lie algebra. It is the maximal solvable subalgebra, a so called Borel subalgebra, of the simple Lie algebra of type A_1.
- The only remaining possibility, such that L does not contain infinite sequences of integers, is $m = \pm n + 1$, $n = 1$. Here we get the following three multiplications:

$$\begin{aligned} [D_1, D_{-n+1}] &= -n \cdot D_{-n+1} \\ [D_1, D_{n+1}] &= n \cdot D_{n+1} \\ [D_{n+1}, D_{-n+1}] &= -2n \cdot D_1 \end{aligned}$$

  These are the multiplications in the three dimensional, simple Lie algebra of type A_1, i.e. $\mathfrak{sl}_2 \cong \mathfrak{su}_2$.

All other combinations lead to infinite dimesional Lie algebras.

3. Solve $\sum_{k=1}^{\infty} \dfrac{1}{k\binom{2k}{k}}$.

   Reason: Puzzle.

   Solution: The Taylor series for $\arcsin^2 x$ in $|x| < 1$ is given by

$$\arcsin^2 x = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k^2\binom{2k}{k}}$$

   Now applying the differential operator $x\frac{d}{dx}$ on both sides yields

$$2x \arcsin x \, \frac{1}{\sqrt{1-x^2}} = \sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k\binom{2k}{k}}$$

   and thus for $x = \frac{1}{2}$:

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{2k}{k}} = \frac{\pi}{6} \cdot \frac{2}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

4. For a given a real Lie algebra $\mathfrak{g}$, we define

$$\mathfrak{A}(\mathfrak{g}) = \{\, \alpha : \mathfrak{g} \longrightarrow \mathfrak{g} \;:\; [\alpha(X), Y] = -[X, \alpha(Y)] \text{ for all } X, Y \in \mathfrak{g} \,\} \quad (1)$$

the set of *antisymmetric transformations* of $\mathfrak{g}$. Remember that a real Lie algebra is a real vector space equipped with a multiplication for which holds

- (2) anti-commutativity: $[X, X] = 0$
- (3) Jacobi-identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Reason: Practice terminology.

(a) Show that $\mathfrak{A}(\mathfrak{g}) \subseteq \mathfrak{gl}(g)$ is a Lie subalgebra in the Lie algebra of all linear transformations of $\mathfrak{g}$ with the commutator as Lie product: $[\alpha, \beta] = \alpha\beta - \beta\alpha \quad (4)$.

Solution:

$$\begin{aligned} [[\alpha, \beta]X, Y] &\overset{(4)}{=} [\alpha\beta X, Y] - [\beta\alpha X, Y] \\ &\overset{(1)}{=} [X, \beta\alpha Y] - [X, \alpha\beta Y] \\ &\overset{(4)}{=} [X, [\beta, \alpha]Y] \\ &\overset{(2)}{=} -[X, [\alpha, \beta]Y] \end{aligned}$$

(b) Give an example of a non Abelian Lie algebra $\mathfrak{g}$ with trivial center, such that $\mathfrak{A}(\mathfrak{g}) \neq 0$.

Solution: $\mathfrak{g} = \langle X, Y \;:\; [X, Y] = Y \rangle$ is non Abelian with trivial center and $\mathfrak{A}(\mathfrak{g}) \cong \mathfrak{sl}(2, \mathbb{R})$. $\mathfrak{g} = \mathfrak{B}(\mathfrak{sl}(2, \mathbb{R}))$ is the maximal solvable subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, a so called Borel subalgebra.

(c) Show that $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$ is a semidirect product given by

$$[X, \alpha] := [\operatorname{ad} X, \alpha] = \operatorname{ad} X\, \alpha - \alpha \operatorname{ad} X \quad (5)$$

Solution: We have to show that this multiplication makes $\mathfrak{A}(g)$

an ideal in $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$ and a $\mathfrak{g}-$module.

$$\begin{aligned}
[[X,\alpha]Y,Z] &\stackrel{(5)}{=} [[X,\alpha Y],Z] - [\alpha[X,Y],Z] \\
&\stackrel{(3),(1)}{=} -[[\alpha Y,Z],X] - [[Z,X],\alpha Y] + [[X,Y],\alpha Z] \\
&\stackrel{(3),(1)}{=} [[Y,\alpha Z],X] + [\alpha[Z,X],Y] \\
&- [[Y,\alpha Z],X] - [[\alpha Z,X],Y] \\
&\stackrel{(2)}{=} [Y,\alpha[X,Z]] - [Y,[X,\alpha Z]] \\
&\stackrel{(5)}{=} -[Y,[X,\alpha Z]]
\end{aligned}$$

and $\mathfrak{A}(\mathfrak{g})$ is an ideal in $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$. It is also a $\mathfrak{g}-$module, because ad is a Lie algebra homomorphism (6) and therefore

$$\begin{aligned}
[[X,Y],\alpha] &\stackrel{(5)}{=} [\operatorname{ad}[X,Y],\alpha] \\
&\stackrel{(6)}{=} [[\operatorname{ad}X,\operatorname{ad}Y],\alpha] \\
&\stackrel{(3)}{=} -[[\operatorname{ad}Y,\alpha],\operatorname{ad}X] - [[\alpha,\operatorname{ad}X],\operatorname{ad}Y] \\
&\stackrel{(2)}{=} [\operatorname{ad}X,[\operatorname{ad}Y,\alpha]] - [\operatorname{ad}Y,[\operatorname{ad}X,\alpha]] \\
&\stackrel{(5)}{=} [X,[Y,\alpha]] - [Y,[X,\alpha]]
\end{aligned}$$

(d) Show that for all $\alpha \in \mathfrak{A}(\mathfrak{g})$ and $X,Y,Z \in \mathfrak{g}$

$$[\alpha(X),[Y,Z]] + [\alpha(Y),[Z,X]] + [\alpha(Z),[X,Y]] = 0 \quad (7)$$

Solution:

$$\begin{aligned}
[\alpha(X),[Y,Z]] &\stackrel{(3)}{=} -[Y,[Z,\alpha(X)]] - [Z,[\alpha(X),Y]] \\
&\stackrel{(1)}{=} [Y,[\alpha(Z),X]] + [Z,[X,\alpha(Y)]] \\
&\stackrel{(3)}{=} -[\alpha(Z),[X,Y]] - [X,[Y,\alpha(Z)]] \\
&- [X,[\alpha(Y),Z]] - [\alpha(Y),[Z,X]] \\
&\stackrel{(1)}{=} -[\alpha(Y),[Z,X]] - [\alpha(Z),[X,Y]]
\end{aligned}$$