Chapter 1

The Scientific Method

PHYSICS deals with describing and interpreting natural phenomena using the *scientific method* which proceeds through the following steps:

① **Schematization:** To reduce the complexity of phenomena, considering the main elements in the first place and everything else as a *perturbation*. This process leads to a *simplified model* of the natural phenomenon.

② **Measurement:** A sequence of procedures to associate a number with unit of measurement (Einheit) to a *phyiscal quantity*. Any physical quantity must be defined in an operational manner, i.e. precise and universally accepted rules must be given that allow it to be measured.

③ **Experimental Observation** of quantitative correlations between numerical values of the measurement of the physical quantities involved in the phenomenon. These correlations are typically presented as tables, graphs, or mathematical formulas.

④ The organization of the results of the experimental observation in **laws** governing the phenomenon. These are typically expressed as mathematical relations between the different physical quantities involved in the phenomenon.

⑤ **Prediction** of new phenomena.

⑥ **Experimental Verification** of the predictions.

1.1 Operational Definition of Physical Quantities

We can define a physical quantity if we have specified the way in which it can be *measured* (i.e. expressed by a number). An important condition is that it must be *reproducible*.
The measurement can be carried out in a *direct* or *indirect* way. To define a *direct measurement*, one has to:

1. Establish a criterion for comparison

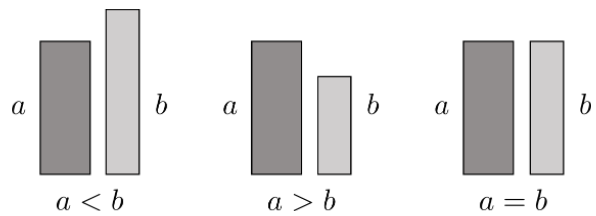


Figure 1.1: Criterion for comparison for two quantities a and b.

2. Define the summation criterion

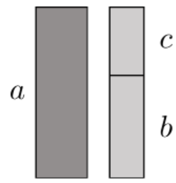


Figure 1.2: Criterion for summation for a quantity a equal to the sum of two quantities b and c.

3. Choose the unit of measurement (or *sample*) for the physical quantity we are considering.

1.2 Errors in the Measurements

If a quantity is measured n times, we usually obtain slightly different values. These fluctuations are due to random errors, which have different origins.

Definition 1.1 *The measurement of a physical quantity Q has a **standard error** σ if there is a $\sim 30\%$ probability that the results of the measurement fall within an interval 2σ.*

Example 1.1 *Let us consider the following distribution of the length of some object.*

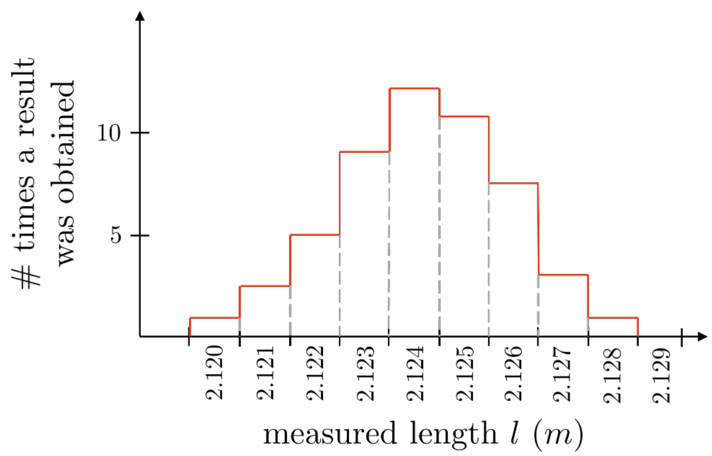


Figure 1.3: Example distribution: Statistical error of a quantity. Here, the length l of some object.

70% of the results fall in the interval $\Delta l = 4$ mm, $\sigma = \frac{\Delta l}{2} = 2$ mm. We should write the value associated to this measurement as:

$$l \pm \sigma = (2.124 \pm 0.002) \text{ m}.$$

Definition 1.2 *If we perform n measurements, we usually consider the* ***mean value*** $\bar{q}$ *as the representative one:*

$$\bar{q} = \frac{q_1 + q_2 + q_3 + \ldots + q_n}{n} = \frac{\sum_{i=1}^{n} q_i}{n}.$$

1.3 System of Units

Definition 1.3 *A quantity whose unit is "arbitrarily" chosen is a* ***fundamental quantity****.*

$$\text{length} \longrightarrow \text{m}$$

Definition 1.4 *If instead its unit is bound to the units of other quantities, then it is called a* ***derived quantity****.*

$$\text{area} \longrightarrow \text{length} \cdot \text{length} \longrightarrow \text{m}^2$$
$$[A] = [L][L] = [L^2]$$

This is a *dimensional equation.* Any physical equation is *dimensionally homogeneous*: the dimensions (units) of the two members of any physical equation must be equal to each other.

1.3.1 International System of Units

	Unit	Abbreviation	Physical Quantity
base units	meter	m	length
	second	s	time
	kilogram	kg	mass
	ampere	A	electric current
	kelvin	K	thermodynamic temperature
	candela	cd	luminous intensity
	mole	mol	amount of substance.

Table 1.1: Base quantities in the international system of units.

Every other physical quantity can be expressed as a function of the base units:

$$\text{acceleration: } [L][t^{-2}] \qquad \text{force: } [M][L][t^{-2}] \qquad \text{density: } [M][L^{-3}]$$

1.3.2 Order of Magnitude

Name	Symbol	Base 10		
yotta	Y	10^{24}	septillion/quadrillion	
zetta	Z	10^{21}	sextillion/trilliard	cosmology
exa	E	10^{18}	quintillion/trillion	
				10^{16} astrophysics
peta	P	10^{15}	quadrillion/billiard	
tera	T	10^{12}	trillion/billion	
giga	G	10^{9}	billion/milliard	
mega	M	10^{6}	million	
kilo	k	10^{3}	thousand	geophysics
hecto	h	10^{2}	hundred	
deca	da	10^{1}	ten	
		10^{0}	one	classical physics
deci	d	10^{-1}	tenth	
centi	c	10^{-2}	hundredth	
milli	m	10^{-3}	thousandth	
micro	μ	10^{-6}	millionth	biophysics
nano	n	10^{-9}	billonth/millardth	atom & solid state physics
pico	p	10^{-12}	trillionth/billionth	
femto	f	10^{-15}	quadrillionth/billiardth	nuclear & particle physics
atto	a	10^{-18}	quintillionth/trillionth	
zepto	z	10^{-21}	sextillionth/trilliardth	
yocto	y	10^{-24}	septillionth/quadrillionth	

Table 1.2: Decimal prefix and order of magnitude.

1.3.3 Scalar and Vector Quantities

Definition 1.5 ***Scalars*** *are quantities that are fully described by a magnitude (or numerical value) alone (e.g. length, mass, temperature).*

Definition 1.6 ***Vectors*** *are quantities that are fully described by both a magnitude and a direction (e.g. velocity, acceleration, force).*

Vectors are commonly denoted in bold font $\mathbf{v}$ or by arrows on top $\vec{v}$. We will use the arrow option to make the distinction easier when writing the lecture notes by hand.

1.4 Relationship Between Dependent and Independent Variables

Definition 1.7 *Often, physical quantities involved in a phenomenon are not constant and their value can change either spontaneously or as a consequence of another parameter (temperature, pressure, ...).*
*These quantities are called **variables**.*

*Often, a variable changes its value as a consequence of the change of the value of another variable. In this case, we say that a variable is a **function** of another one.*

A variable y is a function of another variable x if there is a law expressing the correspondence between the value of y and the value of x.

$$\underset{\text{dependent variable}}{y} = f(\underset{\text{independent variable}}{x}) \qquad \text{or} \qquad y = y(x)$$

A function is said to be **univocal** if for each value of x there exists one and only one value of y.

1.4.1 Table Representation

We can represent a pair of variables using a table with two columns (next to each other) where the values of the independent and dependent variables are given.

Quantity Units	Time t (date and time)		Temperature T ($^\circ C$)	
	22/09	03:00	5	
	22/09	06:00	3	$+2$
	22/09	09:00	7	$+4$
	22/09	12:00	14	$\vdots$
	22/09	15:00	18	
	22/09	18:00	18	
	22/09	21:00	12	
	22/09	24:00	8	

Table 1.3: Table representation of dependent variables. Here, we consider the temperature T depending on the date and time. The time t is the independent quantity.

1.4.2 Graphical Representation

The values of a variable x can be plotted on an oriented straight line (axis). We should also define an origin $\mathcal{O}$ of the axis corresponding to the zero value of the variable.

If we choose two orthogonal axes with their origin at the crossing point, we can establish a bi-univocal correspondence between the points on the plane identified by the two axes and the coordinates of the two variables.

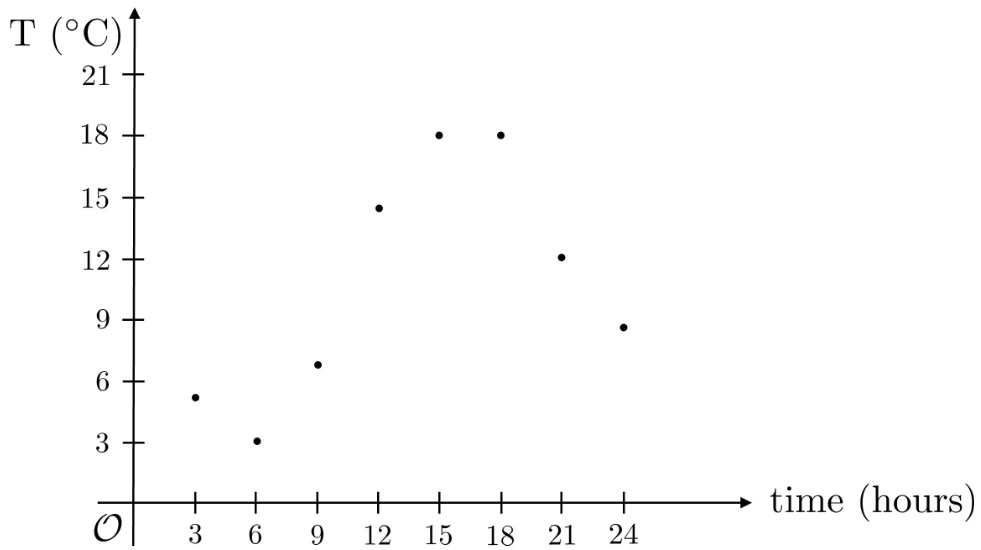


Figure 1.4: Graphical representation of dependent variables. This graph displays the values presented in the table above.

Chapter 2

Kinematics

The word *kinematics* originates from the Greek word κινημα (kinema := motion) from κινειν (“to move“).

Kinematics is simply the mathematical description of motion and makes no reference to the *force* that caused the motion. The simplest system that can be considered in this respect is a *point mass* (Massenpunkt).

We can schematize a system as a point mass if its dimensions are negligible with respect to the precision with which we want to determine its position.

e.g. position of a ship in the ocean, planets in the universe...

From a graphical representation point of view, we can depict a point mass as a geometrical point.
Therefore, the kinematics of a point mass aims at describing the variation of the position of a point mass as a function of time.

2.1 Position and Trajectories

2.1.1 Position

We know the position of a body when we know where it is. Position is a *relative concept* and we need a reference system.

The position $P(t)$ of a point mass in a three dimensional space can be described by its coordinates, which are defined if a suitable coordinate system is chosen.

$$\begin{aligned} &\{x, y, z\} && \text{in a Cartesian system} \\ &\{r, \theta, \varphi\} && \text{in a spherical coordinate system} \\ &\{\rho, \theta, z\} && \text{in cylindrical coordinates} \end{aligned}$$

If the points in space are in a bi-univocal correspondence with an ordered triplet of numbers, we say that space has three dimensions. Similarly, a plane has two dimensions while points on a line defines a one-dimensional space.

Depending on the dimension of space, the coordinate systems are different.

In the plane, there are only two axes needed to define the position.

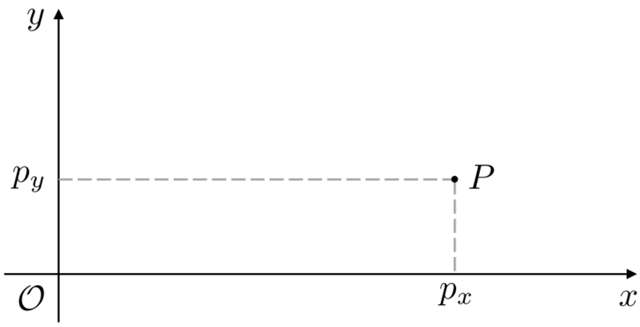
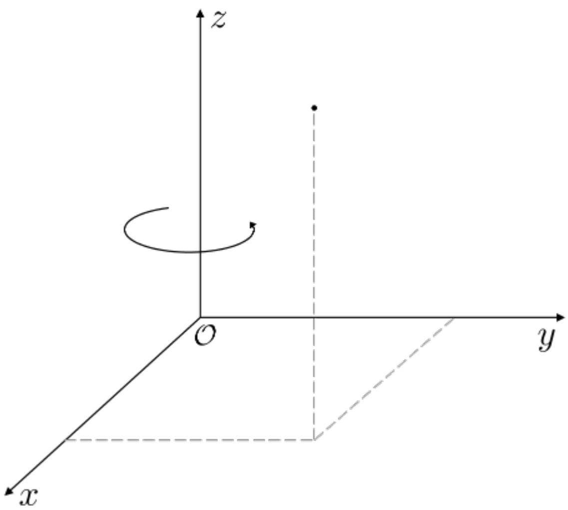
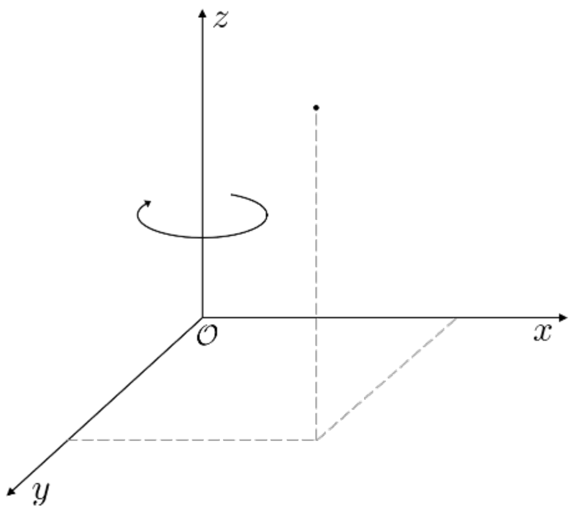


Figure 2.1: System of orthogonal coordinate axes. The point P has two coordinates: p_x and p_y, therefore: $P = (p_x, p_y)$.

In three-dimensional space, we need three axes. Depending on their orientation, we speak of *right-* or *left-handed* coordiante systems.



(a) Right-handed coordinate system.



(b) Left-handed coordinate system..

Figure 2.2: Different orientations of coordinate systems.

2.1.2 Degrees of Freedom

Definition 2.1 *The number of independent free parameters needed to identify the position of a system is called **degree of freedom** of that system.*

A point mass moving freely in *space* has *three* degrees of freedom, a point mass moving freely on a *plane* has *two* degrees of freedom.
The number of degrees of freedom does *not* coincide with the dimensions of the space in which the system is moving.

The number of degrees of freedom of an “extended“ system (not a point mass) is typically bigger than the dimensions in which the system is moving.
If a system does not move freely but is constrained, the number of degrees of freedom decreases.

Example 2.1 *Consider a system consisting of N mass points that can move freely in space, independently of each other. How many degrees of freedom does the system possess?*

Solution: For each point mass, there are 3 degrees of freedom, therefore, the system itself has $3N$ degrees of freedom.

Example 2.2 *A system consists of a rigid rod of negligible cross section and length L. It can move freely in space. How many degrees of freedom does the system possess?*

Solution: We consider the following schematic:

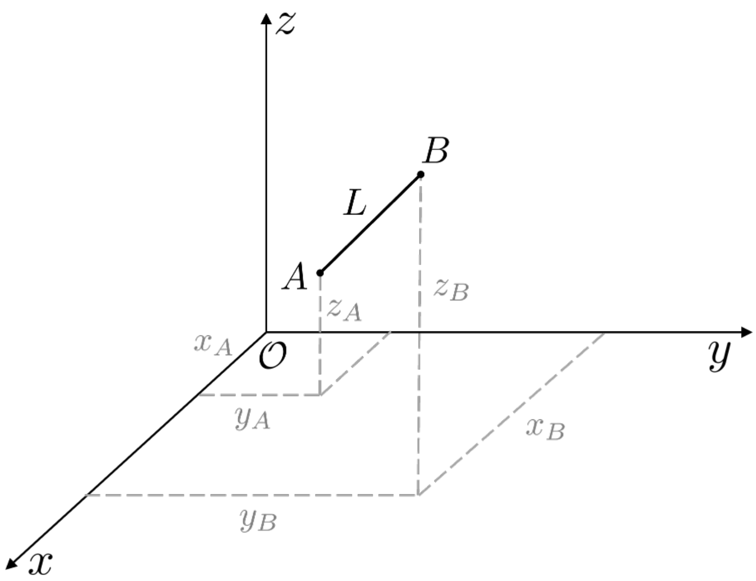


Figure 2.3: Schematic for the example.

The position of the ends of the rod, A and B, is given by

$$A = (x_A, y_A, z_A) \qquad\qquad B = (x_B, y_B, z_B)$$

But the six parameters are *not* independent of each other:

$$L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

Therefore, we can express one of the six parameters as a function of the others, and the degrees of freedom of the system are 5.

2.1.3 Trajectory

Definition 2.2 *The trajectory is given by all the points in space that a point mass occupies during its motion.*

Any point mass on a *predetermined* trajectory has only one degree of freedom.

The motion of a point mass is described as the change of its coordinates with time.

$$\left.\begin{array}{c} x = x(t) \\ y = y(t) \\ z = z(t) \end{array}\right\} \vec{r} = \vec{r}(t) \qquad \text{with } \vec{r} = (x, y, z) \text{ the } \textit{position vector}$$

The function $\vec{r}(t)$ represents the trajectory of the point in a three-dimensional space as a function of time.

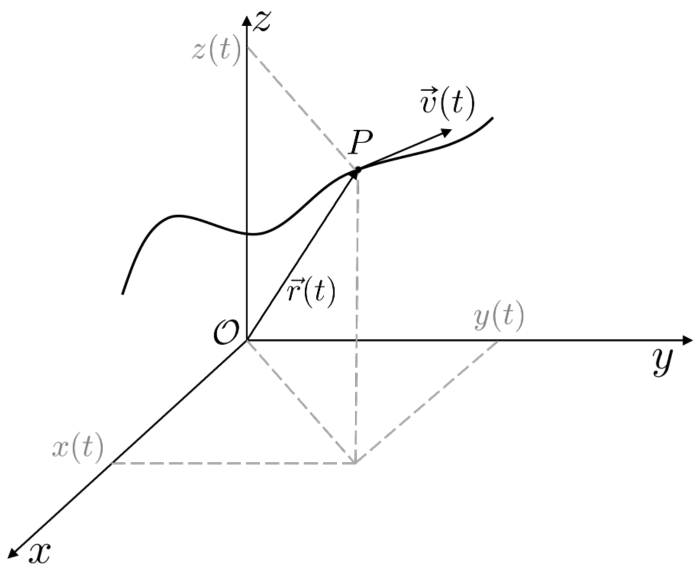


Figure 2.4: Parameter representation of a trajectory. At any instant t, the position vector $\vec{r}(t)$ points to a dot on the trajectory. Over time, it sweeps the total trajectory of the point.

$\vec{r}(t)$ is a *parameter representation* because the coordinates of the point $P(t)$ depend on the parameter t.

Definition 2.3 *The motion performed by $P(t)$ on its trajectory is called* ***translation****. Contrary to the point mass, bodies with extended size can also perform rotations and vibrations.*

Example 2.3 *Motion on a straight line:*
A point mass moves as a function of time according to the representation:

$$x = 0$$
$$y = bt + c \qquad (t \geq 0) \qquad \textit{with } b, c \textit{ constants.}$$
$$z = 0$$

Discuss the motion.

Solution: For any value of t, $x = z = 0 \Longrightarrow$ the point is confined on the y-axis, which is also its trajectory. It is also called **linear motion**.

For $t = 0$, $y = c$.
The parameter c, which is then a "length", represents the coordinate of the point mass at $t = 0$. The *initial position* of the point mass is $\vec{r}_0 = \vec{r}(0)$, therefore

$$r_{0x} = x(0) = 0, \qquad r_{0y} = y(0) = c, \qquad r_{0z} = z(0) = 0.$$

Starting from $t = 0$, the coordinate y increases linearly with time. If we consider $t_1 < t_2$, at t_1, $y_1 = bt_1 + c$ while at t_2, $y_2 = bt_2 + c$.

The space covered in the time interval $\Delta t = t_2 - t_1$ is given by $\Delta y = y_2 - y_1 = b(t_2 - t_1) = b\Delta t$. Therefore, Δy increases proportionally to Δt and the motion is said **uniform**.

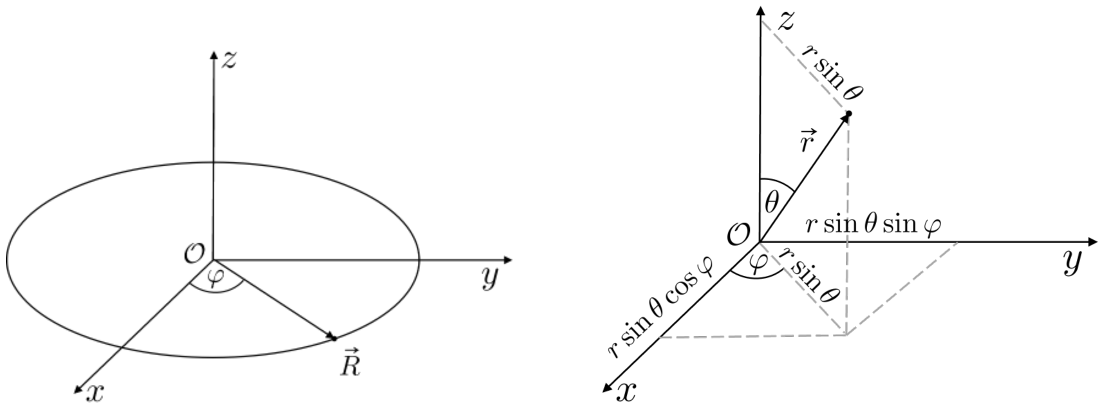
The parameter $b = \dfrac{\Delta y}{\Delta t}$ has the units of a length divided by time: $[L][t^{-1}]$ and represents the space covered in a unit of time: the *velocity*.

Example 2.4 *Uniform circular motion:*
*A point mass is constrained to move on a circumference of radius R that lies on the plane xy with center in the origin. The point mass covers equal arcs in equal times (**uniform circular motion**).*
Write down the equation of motion in the Cartesian coordinates as well as in the polar coordinate system.

Solution: As the trajectory lies in the xy plane, $\theta = \frac{\pi}{2}$. Furthermore, $r = \text{const.} = R$.
Since the point mass covers equal arcs s in equal times:

$$\varphi = \frac{s}{r} = \frac{vt}{R} \qquad \text{with } v \text{ constant.}$$

The origin of time has been chosen such that $t = 0$ when the point mass crosses the x-axis.



(a) Trajectory in Cartesian coordinates. **(b)** Conversion to spherical coordinates.

Figure 2.5: Representation of the circular trajectory of the point mass.

Therefore, the equation of motion in spherical coordinates is:

$$r = R \qquad \theta = \frac{\pi}{2} \qquad \varphi = \frac{vt}{R}.$$

v is the arc covered in a unit of time. Since v is constant, the motion is **uniform**.

The quantity $\frac{v}{R} := \omega$ (also constant), which is the angle φ covered in a unit of time, is called **angular velocity**

$$\omega = \frac{\Delta\varphi}{\Delta t}.$$

The equation of motion in the Cartesian coordinates is:

$$x = r \sin\theta \cos\varphi = R \cos\omega t \qquad y = r \sin\theta \sin\varphi = R \sin\omega t \qquad z = r \cos\theta = 0.$$

NB: as the point mass is constrained to move on a specific trajectory, it has only one degree of freedom.

(This is because in the spherical system, r and θ are constant; in the Cartesian system x and y are variables *but* they must satisfy $x^2 + y^2 = R^2$.)

A planar uniform circular motion is an example of a *periodic motion.* The time needed to cover an entire cycle is called period T.
At $t,\ t+T,\ t+2T,\ ...,\ t+nT$ the point mass is always in the same position with the same kinematic characteristics.
In the case of a uniform circular motion

$$\omega = \frac{\Delta\varphi}{\Delta t} = \frac{2\pi}{T} \qquad \text{or} \qquad T = \frac{2\pi}{\omega}.$$

Experiment 1 *The periodicity of a uniform circular motion can be demonstrated with the experiment* ***harmonische Schwingung****.*

2.2 Velocity and Acceleration

2.2.1 Average Velocity

Definition 2.4 *The* ***velocity*** *is the physical quantity that indicates "how fast" a point mass is moving. Its unit is* $[v] = 1\ m/s$.

If we know the equation of motion, we can calculate the velocity.

We consider a point mass moving with the equation of motion

$$\vec{r} = \vec{r}(t)$$

or in the Cartesian representation

$$r(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

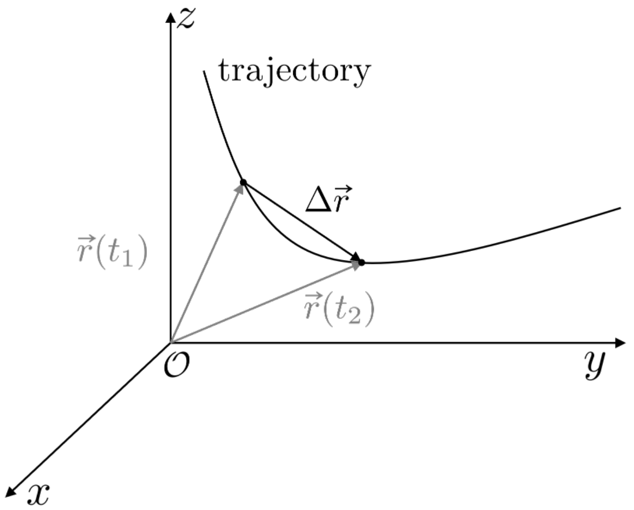


Figure 2.6: In the time interval $\Delta t = t_2 - t_1$, the position vector $\vec{r}(t)$ has covered a distance $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$.

Definition 2.5 *We can define the* ***average velocity*** *in the interval of time* $\Delta t = t_2 - t_1$*:*

$$\bar{\vec{v}}(t_1, t_2) = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

Since the average velocity is the ratio between a vector quantity and a scalar, it is itself a vector quantity.

$$\vec{\bar{v}} = \begin{cases} \bar{v}_x = \dfrac{\Delta r_x}{\Delta t} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1} \\ \bar{v}_y = \dfrac{\Delta r_y}{\Delta t} = \dfrac{y(t_2) - y(t_1)}{t_2 - t_1} \\ \bar{v}_z = \dfrac{\Delta r_z}{\Delta t} = \dfrac{z(t_2) - z(t_1)}{t_2 - t_1} \end{cases}$$

The components of the average velocity represent the average velocity with which the *projections* of the point mass move along the axes.

2.2.2 Momentary Velocity

Definition 2.6 *For $\Delta t \to 0$, $\Delta\vec{r}$ becomes infinitesimally small and we can define the* ***momentary velocity*** *$\vec{v}(t)$ as the limiting value*

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}}$$

which is the time derivative of the function $\vec{r}(t)$.

The momentary velocity vector has, at any point, the direction of the tangent of the trajectory, as $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ gives the slope of the curve $f(x)$.

$$\text{If} \qquad \vec{r}(t) = (x(t), y(t), z(t)), \qquad \text{then} \qquad \vec{v}(t) = (x'(t), y'(t), z'(t)).$$

2.2.3 Acceleration

The velocity itself is function of time and it can change over time. We can therefore wonder how fast it changes over time.

Definition 2.7 *The rate with which the velocity changes is described by the* ***acceleration*** *$\vec{a}(t)$.*

We define the mean acceleration $\vec{\bar{a}}$ as:

$$\vec{\bar{a}} = \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1}$$

The momentary acceleration is the limit:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t).$$

The acceleration is the first time derivative $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ of the velocity $\vec{v}(t)$ and the second derivative $\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}$ of the position vector $\vec{r}(t)$.

$$\vec{a}(t) = (v'_x(t), v'_y(t), v'_z(t)) = (x''(t), y''(t), z''(t))$$

and it has dimensional units $[a] = [m][s^{-2}]$.

2.3 Uniformly Accelerated Motion

We call a motion *uniformly accelerated* for $\vec{a} = \text{constant}$ (i.e. both magnitude and direction of the acceleration do not change with time):

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \vec{a} = \text{constant}$$

This is a differential equation. It links the derivative of a function to another quantity.
We can also write:

$$\ddot{x}(t) = a_x \qquad \ddot{y}(t) = a_y \qquad \ddot{z}(t) = a_z.$$

2.3.1 From Acceleration to the equation of motion

If we know the components of the acceleration $\vec{a} = \vec{a}(t)$ of a point mass

$$a_x = a_x(t) \qquad a_y = a_y(t) \qquad a_z = a_z(t)$$

we can determine the velocity integrating the acceleration:

$$v_x = \int a_x(t)\,\mathrm{d}t + c_x$$
$$v_y = \int a_y(t)\,\mathrm{d}t + c_y$$
$$v_z = \int a_z(t)\,\mathrm{d}t + c_z$$

where $\vec{c} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$ can be determined choosing the *initial conditions* for the motion:

$$\text{for } t = 0: \;\; \dot{\vec{r}}(0) = \vec{v}(0) = \vec{c}$$

We can further integrate the expression of the velocity to obtain the equation of motion.

Example 2.5 *Let us calculate the equation of motion in the case of a uniform acceleration.*

Solution: In the case of a uniformly accelerated motion:

$$\vec{v}(t) = \dot{\vec{r}}(t) = \int \vec{a}\,\mathrm{d}t = \vec{a}t + \underbrace{\vec{v}_0}_{\vec{v}(t=0)}$$

and we can calculate the trajectory as:

$$\vec{r}(t) = \int (\vec{a}t + \vec{v}_0)\,\mathrm{d}t = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{c} \qquad \text{with} \qquad \vec{c} = \vec{r}(t=0) = \vec{r}_0$$

We can also write the three components:

$$\begin{cases} x(t) &= \frac{1}{2}a_x t^2 + v_{0x}t + x_0 \\ y(t) &= \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \\ z(t) &= \frac{1}{2}a_z t^2 + v_{0z}t + z_0 \end{cases}$$

2.3.2 Free Fall (With Experiment)

Example 2.6 *We consider the free fall motion of a point mass according to the figure below.*

Solution:

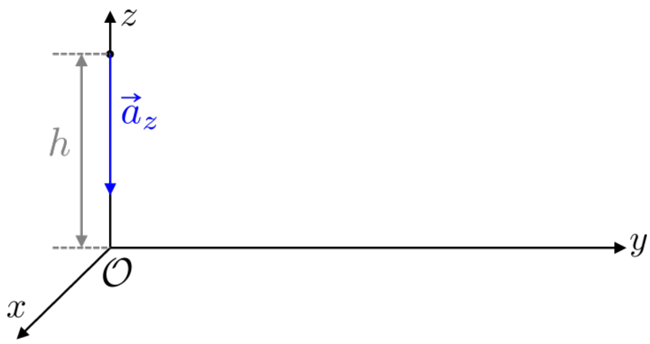


Figure 2.7: A point mass, at height h above ground, is initially at rest and subject to an acceleration $\vec{a}_z$ pointing in the $-\hat{z}$ direction.

For the components of the acceleration:

$$a_x = a_y = 0 \qquad\qquad a_z = -g = -9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

As the body is starting at rest, at $t = 0$: $\vec{v} = \vec{0}$. Its initial position is given by:

$$\begin{cases} x(0) = 0 \\ y(0) = 0 \\ z(0) = h \end{cases}$$

and the equation of motion writes

$$z(t) = -\frac{1}{2}gt^2 + h.$$

We can calculate the velocity:

$$v_z(t) = -gt.$$

If we plot $z(t)$:

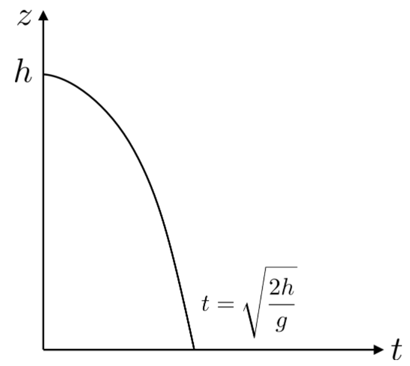


Figure 2.8: The diagram shows a parabolic graph as $z(t) \propto t^2$ ($z(t)$ is proportional to t^2).

The point mass touches the ground for $z = 0$ which corresponds to a time of

$$-\frac{1}{2}gt^2 + h = 0 \qquad \Longrightarrow \qquad t = \sqrt{\frac{2h}{g}}$$

and the velocity at impact is:

$$v\left(t = \sqrt{\frac{2h}{g}}\right) = \sqrt{2gh}.$$

2.3.3 Projectile Motion

Example 2.7 *A point mass is thrown horizontally with velocity $\vec{v}_1$ from a height h. Calculate the time t_1 needed to reach the ground and the projectile range y_1. What would the values of t_2 and y_2 be if $\vec{v}_2 = 2\vec{v}_1$?*

Solution: Choosing a Cartesian coordinate system, the trajectory lies in the yz-plane.

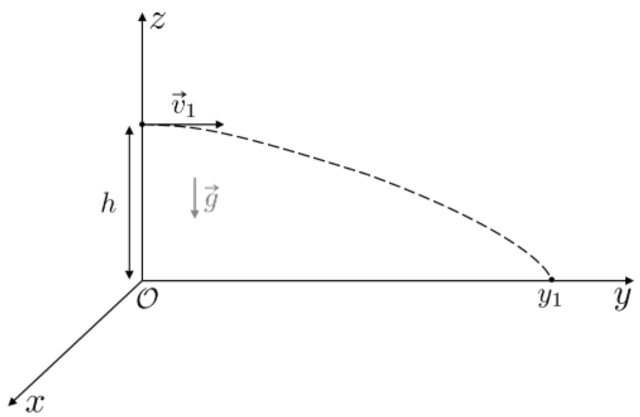


Figure 2.9: Projectile motion (horizontal throw).

We consider the components of the acceleration acting on the point mass:

x-component: $a_x = 0 \Longrightarrow v_x = 0 \Longrightarrow x = 0$
y-component: $a_y = 0 \qquad v_y = \text{const.} = v(t=0) = v_1$

$$\Longrightarrow y(t) = \int v_1 \, \mathrm{d}t + \text{const.} = v_1 t + \underbrace{y(0)}_{=0} = v_1 t$$

z-component: $a_z = -g \Longrightarrow v_z = -gt + \underbrace{v_z(t=0)}_{=0} = -gt$

$$\Longrightarrow z(t) = -\frac{1}{2}gt^2 + z(t=0) = -\frac{1}{2}gt^2 + h$$

t_1 is such that $z(t_1) = 0 \Longrightarrow -\frac{1}{2}gt_1^2 + h = 0 \Longrightarrow t_1 = \sqrt{\frac{2h}{g}}$.
The projectile range is $y(t = t_1):\ y_1 = v_1 \cdot \sqrt{\frac{2h}{g}}$.

If $v_2 = 2v_1$, t_2 is still $\sqrt{\frac{2h}{g}}$ while $y_2 = 2v_1\sqrt{\frac{2h}{g}} = 2y_1$.

Along y: The motion is *uniform*
Along z: The motion is *uniformly accelerated*

The projectile range is proportional to its initial velocity.

2.4 Motions With Non-Constant Acceleration

Example 2.8 *Let us consider a* ***uniform circular motion*** *of a point mass on a circular trajectory of fixed radius R.*

Solution:

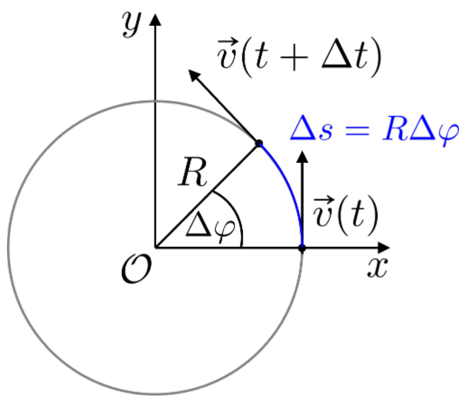


Figure 2.10: Uniform circular motion.

The magnitude of the velocity $|\vec{v}|$ remains constant. Furthermore,

$$\vec{v}(t) = R\omega\ \hat{t}(t), \qquad \vec{r} = R\ \hat{r}(t)$$

where $\hat{t}$ is a unit vector *tangential* to the circular trajectory and $\hat{r}$ is the unit vector pointing from $\mathcal{O}$ to a point on the circle.

The velocity is:

$$\dot{\vec{r}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}R}{\mathrm{d}t}}_{\substack{=0,\text{ as}\\ R\text{ const.}}}\ \hat{r}(t) + R\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = R\omega\ \hat{v}$$

where

$$\omega(t) = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{s}{R}\right) = \frac{1}{R}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v(t)}{R}.$$

We can now derive this expression to obtain the acceleration:

$$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(R\omega\ \hat{v}) = R\frac{\mathrm{d}\omega}{\mathrm{d}t}\ \hat{v} + R\omega\frac{\mathrm{d}\hat{v}}{\mathrm{d}t}$$

The derivative of a unit vector $\hat{e}$ is always perpendicular to the unit vector itself: $\frac{\mathrm{d}\hat{e}}{\mathrm{d}t} \perp \hat{e}$. Take the equation $\hat{e}^2 = 1$ and derive both sides:

$$2\hat{e}\cdot\frac{\mathrm{d}\hat{e}}{\mathrm{d}t} = 0 \Longleftrightarrow \hat{e}\cdot\frac{\mathrm{d}\hat{e}}{\mathrm{d}t} = 0 \Longrightarrow \frac{\mathrm{d}\hat{e}}{\mathrm{d}t} \perp \hat{e}.$$

The term $\frac{\mathrm{d}\hat{v}}{\mathrm{d}t}$ gives the angular velocity of the tangent to the circle: $|\frac{\mathrm{d}\hat{v}}{\mathrm{d}t}| = \omega$.

$$\vec{a}(t) = R\frac{\mathrm{d}\omega}{\mathrm{d}t}\ \hat{v} + R\omega^2\ \hat{n} = \underbrace{R\frac{\mathrm{d}\omega}{\mathrm{d}t}\ \hat{v}}_{\substack{=0\text{ for uniform}\\ \text{circular motion}}}\ - R\omega^2\ \hat{r}.$$

Another way of seeing that $\vec{a}$ is directed towards the center of the circle is considering that the components of $\vec{a}$ are proportional to the components of $\vec{r}$ but with *opposite* sign.

$$\vec{r}(t) = \begin{cases} x = R\cos\omega t \\ y = R\sin\omega t \\ z = 0 \end{cases} \qquad \vec{v}(t) = \begin{cases} \dot{x}(t) = -\omega R\sin\omega t \\ \dot{y}(t) = \omega R\cos\omega t \\ \dot{z}(t) = 0 \end{cases} \qquad \vec{a}(t) = \begin{cases} \ddot{x}(t) = -\omega^2 R\cos\omega t \\ \ddot{y}(t) = -\omega^2 R\sin\omega t \\ \ddot{z}(t) = 0 \end{cases}$$

Therefore, the acceleration of a *circular motion* has two components:

$$a_t = R\frac{d\omega}{dt} \qquad \text{tangential component } (= 0 \text{ if it is a } \textit{uniform motion})$$

$$a_r = R\omega^2 = v\omega = \frac{v^2}{R} \qquad \text{centripetal acceleration}$$

2.5 Motions on Trajectories With Arbitrary Curvature

If we consider a point mass moving on a plane, at any point of the trajectory, we can define the tangent $\hat{t}$ and the normal $\hat{n}$ to the trajectory.

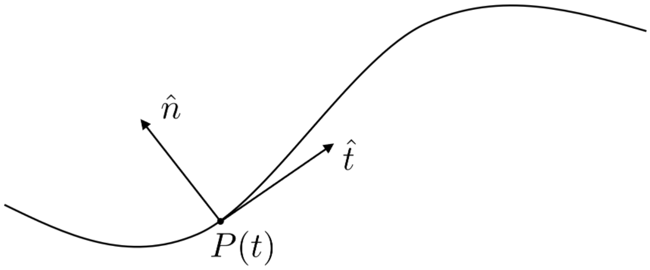


Figure 2.11: Arbitrary curvature. For any point P on the trajectory, $\hat{t}$ and $\hat{n}$ point in the curve's tangential and normal direction, respectively.

The velocity $\vec{v}(t)$ at time t is always the tangent to the trajectory in the point $P(t)$:

$$\vec{v}(t) = v(t)\ \hat{t}.$$

The acceleration instead has both components tangent and normal to the curve.

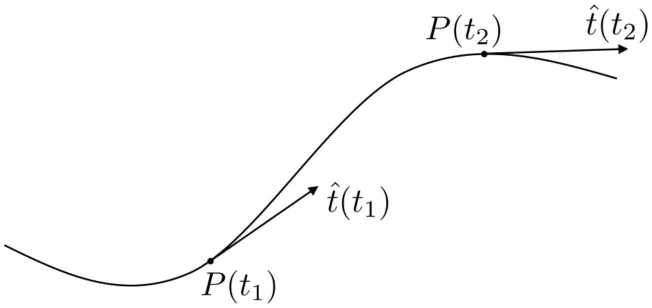


Figure 2.12: Arbitrary curvature. The tangent unit vector $\hat{t}(t)$ changes with respect to time.

The acceleration is

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v(t)\hat{t}) = \underbrace{\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)\hat{t}}_{\substack{\text{tangential}\\\text{component}}} + \underbrace{v\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\right)}_{\substack{\text{normal}\\\text{component}}} = \vec{a}_t + \vec{a}_n$$

$\vec{a}_t$ describes the change in magnitude of velocity, while $\vec{a}_n$ describes the change in the direction of the velocity.

If $a_n = 0$, the trajectory is a straight line with changing velocity if $a_t \neq 0$.
If $a_t = 0$, the body moves on a curve trajectory determined by $\vec{a}_n(t)$ with a constant velocity $|\vec{v}|$.

In the case of the free fall: $a_n = 0$, $a_t = \text{const.}$
In the case of a uniform circular motion: $a_n = \text{const.}$, $a_t = 0$.

Local Radius of Curvature:

$\hat{t}$ and $\hat{n}$ can be decomposed as:

$$\begin{cases} \hat{t} & = \cos\varphi\ \hat{x} + \sin\varphi\ \hat{y} \\ \hat{n} & = \cos\left(\varphi + \frac{\pi}{2}\right)\hat{x} + \sin\left(\varphi + \frac{\pi}{2}\right)\hat{y} = -\sin\varphi\ \hat{x} + \cos\varphi\ \hat{y} \end{cases}$$

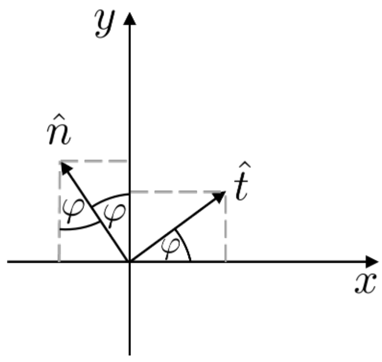


Figure 2.13: Decomposition of $\hat{n}$ and $\hat{t}$.

Deriving the expression for $\hat{t}$:

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}t} = -\sin\varphi\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{x} + \cos\varphi\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{y} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\left(-\sin\varphi\ \hat{x} + \cos\varphi\ \hat{y}\right) = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{n}$$

The *normal acceleration* is therefore:

$$\hat{a}_n = v\ \frac{\mathrm{d}\hat{t}}{\mathrm{d}t} = v\ \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{n}.$$

For an infinitesimally small section of a curve ($A \to A'$), we get $\mathrm{d}s = \rho\,\mathrm{d}\varphi$ where ρ is the radius of curvature.

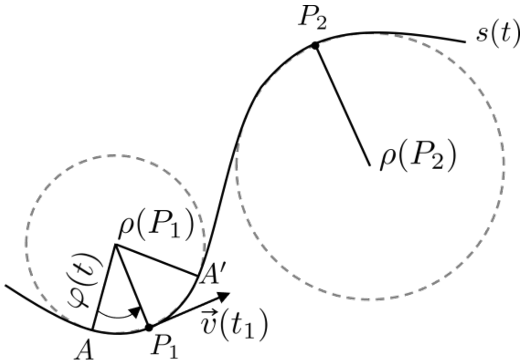


Figure 2.14: Infinitesimal curvature, approximated by two circles of radii $\rho(P_1)$ and $\rho(P_2)$ with P_1 and P_2 points on the trajectory.

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}v = \frac{1}{\rho}v$$

And the acceleration vector becomes

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

Experiment 2 ***Tangentiales Wegfliegen von Kugeln***.

2.6 Summary of Chapter 2

- Kinematics describes the motion of objects without referring to the force which caused the motion.
- We have a full description of the kinematics of a motion when we know the position of the moving object as a function of time.
- We can define the position only within a frame of reference.
- The degrees of freedom define the amount of parameters needed to specify the position of an object.
- The equation of motion is known when we know the position vector as a function of time:

$$\vec{r} = \vec{r}(t) \qquad \text{or} \qquad \vec{r}(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

- The velocity is the vectorial quantity that measures how fast the position changes:

$$\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \qquad \text{average velocity}$$
$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \qquad \text{momentary velocity}$$

  or alternatively:

$$\vec{v}(t) = \begin{cases} v_x = \frac{\mathrm{d}x}{\mathrm{d}t} \\ v_y = \frac{\mathrm{d}y}{\mathrm{d}t} \\ v_z = \frac{\mathrm{d}z}{\mathrm{d}t} \end{cases}$$

- At every instant, the velocity is directed as the tangent to the trajectory.
- The acceleration is the vectorial quantity that measures how fast the velocity $\vec{v}(t)$ of the point mass changes over time:

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \qquad \text{or} \qquad \vec{a}(t) = \begin{cases} a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} \\ a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} \\ a_z = \frac{\mathrm{d}v_z}{\mathrm{d}t} \end{cases}$$

- The acceleration has both a component tangential ($\vec{a}_t$) and one normal ($\vec{a}_n$) to the trajectory:

$$\vec{a} = \vec{a}_t + \vec{a}_n.$$

- The tangential acceleration $\vec{a}_t$ has magnitude equal to the derivative of the magnitude of the velocity

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t}$$

  and it is equal to zero only if the object is moving with a velocity whose magnitude is constant.

- The normal acceleration $\vec{a}_n$ is centripetal (directed towards the center of the circular trajectory that best approximates the actual trajectory locally):

$$a_n = \frac{\vec{v}^2}{R} \qquad \text{with } R \text{ local radius of curvature.}$$

- The velocity $\vec{v}$ and the acceleration $\vec{a}$ can always be calculated when we know the equation of motion.

- Knowing the acceleration $\vec{a}(t)$, we can calculate the velocity $\vec{v}(t)$ given initial conditions:

$$v_x = \int a_x(t)\,\mathrm{d}t + c_{1,x} \qquad v_y = \int a_y(t)\,\mathrm{d}t + c_{1,y} \qquad v_z = \int a_z(t)\,\mathrm{d}t + c_{1,z}$$

- Knowing the velocity $\vec{v}(t)$, we can calculate the equation of motion $\vec{r}(t)$ given initial conditions:

$$r_x = \int v_x(t)\,\mathrm{d}t + c_{2,x} \qquad r_y = \int v_y(t)\,\mathrm{d}t + c_{2,y} \qquad r_z = \int v_z(t)\,\mathrm{d}t + c_{2,z}$$