1.7 BOUNDARY CONDITIONS

The differential equation of equilibrium which must be satisfied within the plate is derived in Sec. 1.6. The distribution of stress in a plate must also be such as to accommodate the conditions of equilibrium with respect to prescribed forces or displacements at the boundary.

For a plate, solution of Eq. (1.17) requires that two boundary conditions be satisfied at each edge. These may be a given deflection and slope, or force and moment, or some combination. The basic difference between the boundary conditions applied to plates and those of beams is the existence along the plate edges of twisting moments. It is demonstrated below that these moments may be replaced by equivalent forces. Such a substitution causes an alteration of the distribution of stress and strain only in the immediate region of the boundary, in accordance with St. Venant's principle.[1]

We now treat the boundary conditions for a rectangular plate with edges a and b parallel to the x and y axes, as shown in Fig. 1.6. Consider two successive elemental lengths dy on edge $x = a$ (Fig. 1.6). It is seen that, on the right-hand element, a twisting moment $M_{xy}\,dy$ acts, while the left-hand element is subjected to $[M_{xy} + (\partial M_{xy}/\partial y)\,dy]\,dy$. In the figure, the moments are indicated as replaced by statically equivalent force couples. Thus in an infinitesimal region of the edge shown within the dashed line, we see that an upward directed force M_{xy} and a downward directed force $M_{xy} + (\partial M_{xy}/\partial y)\,dy$ act. The algebraic sum of these forces may be added to the shearing force Q_x to produce an effective transverse force *per unit length* for an edge parallel to the y axis, V_x. That is

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right] \qquad (1.23a)$$

Similarly, it can be shown that, for an edge parallel to the x axis, one has

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right] \qquad (1.23b)$$

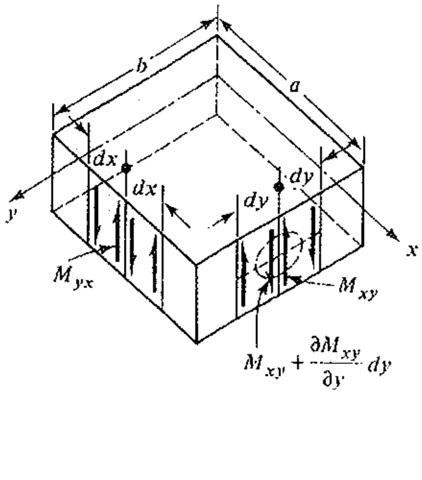

Figure 1.6

Expression (1.23) is due to *Kirchhoff*: a distribution of M_{xy} along an edge is statically equivalent to a distribution of vertical shear forces.

In addition to the edge forces described above, there may be *concentrated forces* F_c produced *at the corners*. Consider, as an example, the case of a *uniformly loaded* rectangular plate with simply supported edges (Fig. 1.6). At the corner (a, b) above-discussed action of twisting moments (because $M_{xy} = M_{yx}$) results in

$$F_c = 2M_{xy} = -2D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} \qquad (x = a,\ y = b) \tag{1.24}$$

The negative sign indicates an upward direction. Owing to the symmetry of the uniform loading, this force must have the same magnitude and direction at all corners of the plate. Thus, if no anchorage is provided, the corners of the plate described tend to rise (Example 3.2).

The additional corner force for plates having various edge conditions may be determined similarly; for instance, when two adjacent plate edges are *fixed* or *free*, we have $F_c = 0$, since along these edges no twisting moment exists.

We can now formulate a variety of commonly encountered situations. The *boundary conditions* which apply along the edge $x = a$ of the rectangular plate with edges parallel to the x and y axes (Fig. 1.7) are as follows.

Clamped or Built-in Edge (Fig. 1.7*a*) In this case both the deflection and slope must vanish. That is

$$w = 0 \qquad \frac{\partial w}{\partial x} = 0 \qquad (x = a) \tag{1.25}$$

Simply Supported Edge (Fig. 1.7*b*) At the edge considered, the deflection and bending moment are both zero. Hence

$$w = 0 \qquad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (x = a) \tag{1.26a}$$

The first of these equations implies that along edge $x = a$, $\partial w/\partial y = 0$,

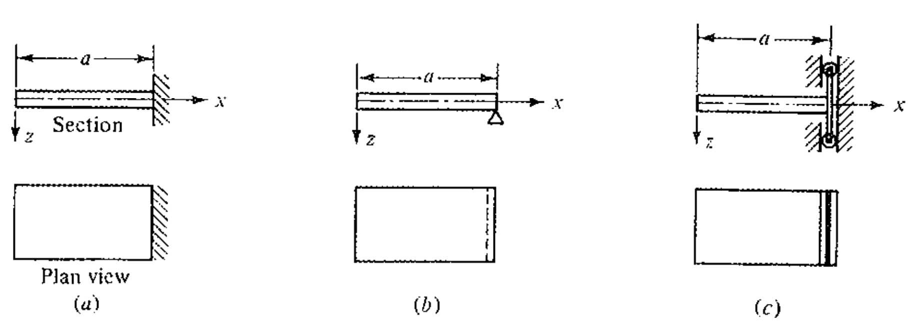

Figure 1.7

$\partial w^2/\partial y^2 = 0$. It follows that conditions expressed by Eqs. (1.26*a*) may appear in the following equivalent form

$$w = 0 \qquad \frac{\partial^2 w}{\partial x^2} = 0 \qquad (x = a) \tag{1.26b}$$

Free Edge Such an edge at $x = a$ is free of moment and vertical shear force. That is

$$\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0 \qquad \frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\,\partial y^2} = 0 \qquad (x = a) \tag{1.27}$$

Sliding Edge (Fig. 1.7*c*) In this case the edge is free to move vertically, but the rotation is prevented. The support is not capable of resisting any shear force. Thus

$$\frac{\partial w}{\partial x} = 0 \qquad \frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\,\partial y^2} = 0 \qquad (x = a) \tag{1.28}$$

Some other types of boundary conditions may be treated similarly. It is observed that the boundary conditions are of two basic kinds: a *geometric* or *kinematic* boundary condition describes end constraint pertaining to deflection or slope; a *static* boundary condition equates the internal forces (and moments) at the edges of the plate to the given external forces (and moments). Accordingly, in Eqs. (1.25) both conditions are kinematic; in (1.27) both are static; in (1.26) and (1.28) the conditions are *mixed*.

In addition to the *homogeneous* boundary conditions described above, it is, of course, possible to have prescribed shear, moment, rotation, or displacement at the boundary. The latter cases, *nonhomogeneous* boundary conditions, are expressed by replacing the zeros in Eqs. (1.25) to (1.28) with the specified quantity (Sec. 9.4).

1.8 METHODS FOR SOLUTION OF PLATE DEFLECTIONS

Except for simple types of loadings and shapes, such as axisymmetrically loaded circular plates (Sec. 2.3), the governing plate equation $\nabla^4 w = p/D$ yields plate deflections only with considerable difficulty. It is common to attempt a solution by the *inverse method*. The inverse method relies upon assumed solutions for w which satisfy the governing equation and the boundary conditions. Some cases may be treated by using polynomial expressions for w in x and y and undetermined coefficients. Usually, choosing the acceptable series form is laborious and requires a systematic approach. The most powerful such method is the *Fourier series*, where, once a solution has been found for sinusoidal loading, any other loading can be handled by infinite series (Secs. 3.2 and 3.4). This approach offers as an important advantage the fact that a *single expression* may apply to *the entire surface* of the plate.