1. Method of Characteristics

The general method of characteristics deals with the following Cauchy problem

$$S_t + H(t, x, S, S_x) = 0, \quad S\mid_{t=0} = \hat{S}(x). \tag{1}$$

And we want to find a solution $S = S(t, x)$.

Here H is a scalar-valued function $H = H(t, x, \xi, p)$;

$$t, \xi \in \mathbb{R}, \qquad x = (x^1, \dots, x^m), \quad p = (p_1, \dots, p_m) \in \mathbb{R}^m;$$

and $S_x = (S_{x^1}, \dots, S_{x^m})$.

In the sequel we will avoid from details such as domains of functions, smoothness conditions etc.

Together with IVP (1) consider a system of ODE:

$$\begin{aligned}
\dot{\xi} &= p_n \frac{\partial H}{\partial p_n} - H, \qquad H = H(t, x, \xi, p);\\
\dot{p}_i &= -\frac{\partial H}{\partial x^i} - \frac{\partial H}{\partial \xi} p_i, \qquad i = 1, \dots, m\\
\dot{x}^i &= \frac{\partial H}{\partial p_i}.
\end{aligned} \tag{2}$$

This system is referred to as the characteristic system.

If H does not depend on ξ then the equations for x, p are separated and turn into a Hamiltonian system while equation (1) turns into the Hamilton-Jacobi equation.

The extended phase space of (2) is as follows

$$M = \{(t, x, \xi, p)\} \subset \mathbb{R}^{2m+2}.$$

The method of characteristics is based upon the following theorem.

Theorem 1. *Let $S = S(t, x)$ be a solution to (1). Then the graph*

$$G = \{\xi = S(t, x), \quad p = S_x(t, x)\} \subset M$$

is an invariant manifold for (2). In other words, the manifold G consists of the trajectories of the characteristic system, $\dim G = m + 1$.

Proof. Let $x(t)$ be a solution to the ODE

$$\dot{x} = \frac{\partial H}{\partial p}(t, x, S(t, x), S_x(t, x)).$$

Let us show that

$$x(t), \quad \xi(t) := S(t, x(t)), \quad p(t) := S_x(t, x(t))$$

is a solution to (2).

Indeed,

$$\dot{\xi} = S_t + S_x \dot{x} = -H + p_i \frac{\partial H}{\partial p_i}(t, x, S(t, x), S_x(t, x)).$$

To proceed introduce a function

$$F(t,x) := H(t,x,S(t,x),S_x(t,x)).$$

Observe that

$$\frac{\partial F}{\partial x^i} = \frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial \xi}\frac{\partial S}{\partial x^i} + \frac{\partial H}{\partial p_s}\frac{\partial^2 S}{\partial x^i \partial x^s}.$$

So we yield

$$\begin{aligned}\dot p_k &= \frac{\partial^2 S}{\partial t \partial x^k} + \frac{\partial H}{\partial p_s}\frac{\partial^2 S}{\partial x^k \partial x^s} = \frac{\partial^2 S}{\partial t \partial x^k} + \frac{\partial F}{\partial x^k} - \frac{\partial H}{\partial x^k} - \frac{\partial H}{\partial \xi}\frac{\partial S}{\partial x^k} \\ &= \frac{\partial}{\partial x^k}\Big(\frac{\partial S}{\partial t} + F\Big) - \frac{\partial H}{\partial x^k} - \frac{\partial H}{\partial \xi}\frac{\partial S}{\partial x^k} = -\frac{\partial H}{\partial x^k} - p_k\frac{\partial H}{\partial \xi}.\end{aligned}$$

The theorem is proved.

This theorem prompts the following method of solving (1).

Let us provide system (2) with initial conditions

$$x\mid_{t=0} = \hat x, \quad \xi\mid_{t=0} = \hat S(\hat x), \quad p\mid_{t=0} = \hat S_x(\hat x).$$

The corresponding solution to (2) is

$$x = X(t,\hat x), \quad \xi = \Xi(t,\hat x), \quad p = P(t,\hat x).$$

To obtain the function $S(t,x)$ one must express $\hat x$ from the equation $x = X(t,\hat x)$ and substitute it to the equation $S = \Xi(t,\hat x)$.

2. Example: The Hopf Equation

Consider the following IVP

$$S_t + \frac{1}{2}(S^2)_x = 0, \quad S\mid_{t=0} = -x \in \mathbb{R}.$$

This is the classical Hopf Equation. It arises in different physics phenomena. For example, it is often interpreted as a baby version of the Euler equation for 1-dimensional perfect compressible fluid's flow.

Here the Hamiltonian is $H = \xi p$ and the characteristics equations are

$$\dot\xi = 0, \quad \dot p = -p^2, \quad \dot x = \xi.$$

Solving this system with initial conditions

$$x(0) = \hat x, \quad \xi(0) = -\hat x, \quad p(0) = -1$$

we obtain

$$x = \hat x(1-t), \quad \xi = -\hat x$$

So that

$$S(t,x) = -\frac{x}{1-t}.$$

We see that the solution to the Hopf equation exists only for $t \in [0,1)$ and for $t = 1$ we get a blowup.

The cause of this effect is as follows. The solutions to the characteristic equations are projected to the plane $\{(t,x)\}$ into the family of straight lines

$$\{x = \hat{x}(1-t) \mid \hat{x} \in \mathbb{R}\}.$$

All these lines intersect at the point $t = 1, \quad x = 0$ and bring different values of $S(t,x)$ from the line of initial condition $\{t = 0\}$ to this point. Thus the solution to the Hopf equation can not be correctly defined at $(1,0)$.

In terms of the previous section we obtain such an effect when the equation $x = X(t,\hat{x})$ can not be solved globally with respect to $\hat{x}$.

Singularities in the initial condition $\hat{S}$ provide another source of blowups.

Observe also that if we impose other initial condition $S\mid_{t=0}= x$ for the Hopf equation then there are no blowups.