An Introduction to the Theory of Characteristics

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Abstract

This document is essentially and introduction to first order quasilinear partial differential equations. We shall approach the solution of such equations via a geometrical approach called the method of characteristics, we shall see that this greatly reduces the complexity of the problem of solution. The end goal is to apply these methods to inviscid fluid flow which will reduce the nonlinear set of PDE's to a set of algebraic equations.

1 Linear 1st Order Partial Differential Equations

For a model problem take the PDE:

$$a(t,x)\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} = c(t,x) \tag{1}$$

Where a, b, c are functions of t and x and the initial condition $u(0,x) = f(x)$. At first sight this seems quite a difficult problem to tackle.

1.1 A Geometrical Approach

The solution of the partial differential equation (2) can be viewed as a two dimensional surface, viewing the problem in this manner allows the use of geometric constructs which allow the problem to be reduced in complexity. To start with the PDE is rewritten in a slightly trivial way as:

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} - c = 0 \tag{2}$$

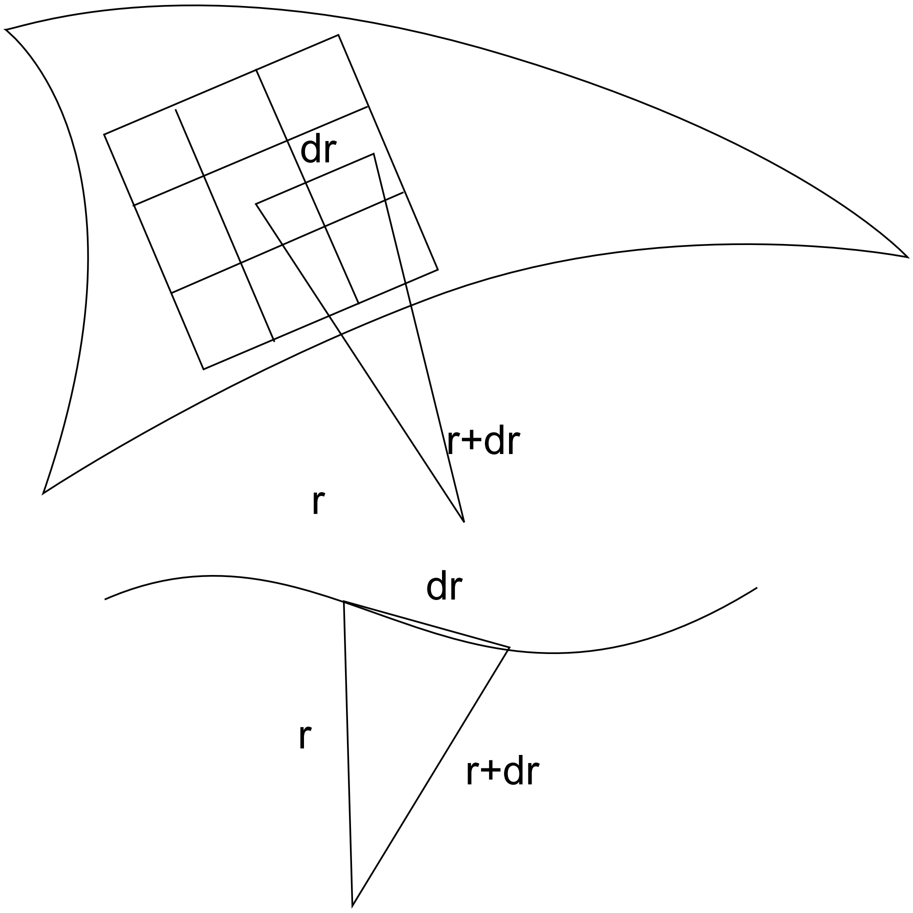

Figure 1: Gradient of a surface

Now write (2) as a dot product of two three dimensional vectors in the following manner:

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \left(\frac{\partial u}{\partial t}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j} - \mathbf{k} \right) = 0 \tag{3}$$

At first sight this doesn't seem to help matters much but if $y = u(t, x)$ is viewed as a two dimensional surface in three dimensional space then the vector:

$$\mathbf{n} := \frac{\partial u}{\partial t}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j} - \mathbf{k}$$

defines a normal to the surface $y = u(t, x)$ at every point. To prove this fact consider the position vector $\mathbf{r} = t\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the vector $\mathbf{r} + d\mathbf{r}$.

Completing the triangle of the vectors as in Figure 1, it can clearly be seen that the vector $d\mathbf{r}$ lies in the tangent plane to the surface.
The three dimensional gradient operator in the variables is:

$$\nabla = \mathbf{i}\frac{\partial}{\partial t} + \mathbf{j}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial u}$$

Writing the equation of the surface as $f(t,x) - u = \phi$ and computing $d\phi$ using the chain rule gives:

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial u}du \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx - du \\ &= \left(\frac{\partial f}{\partial t}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j} - \mathbf{k}\right) \cdot (dt\mathbf{i} + dx\mathbf{j} + du\mathbf{k}) \\ &= \nabla\phi \cdot d\mathbf{r} \\ &= 0 \end{aligned}$$

As $d\mathbf{r}$ lies in the tangent plane, the other vector in the dot product must be at right angles to the tangent vector, that is it must be a normal to the surface.
So, this shows that the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ must be a tangent vector. It is possible to build a curve $(t(s)\mathbf{i} + x(s)\mathbf{j} + u(s)\mathbf{k})$ paramatrised by s on the surface with this tangent vector, these curves are called *characteristics*. So now it is possible to effectively solve the equations on this special curve as the partial differential equation is now written in the following manner:

$$\frac{dt}{ds} = a, \quad \frac{dx}{ds} = b, \quad \frac{du}{ds} = c \tag{4}$$

So The next question is how it is possible to fit the initial data into this set-up.

1.2 Cauchy Data

The notion of Cauchy data is quite simple. Given a curve $\Gamma(v)$ in the (t,x) plane, Cauchy data is simply the prescription of initial data (of u) on that curve. The initial data can be written in the form:

$$t = t_0(v), \quad x = x_0(v), \quad u = u_0(v) \tag{5}$$

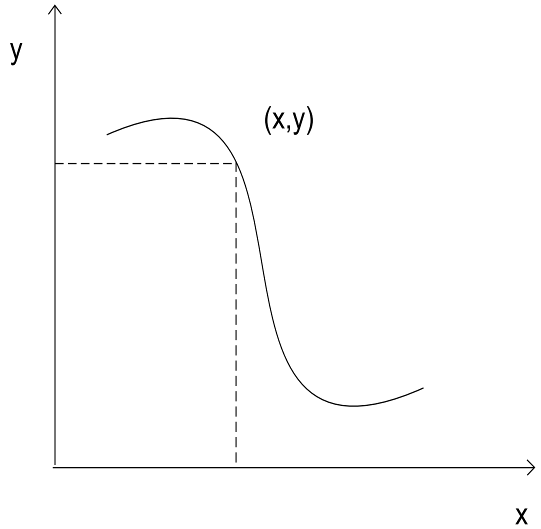

Figure 2: Cauchy Data

Denoting the derivative with respect to s as $dx/ds = \dot{x}$ and the derivative with respect to v $dx/dv = x'$ it is possible to differentiate the initial data along the curve Γ as:

$$u_0' = \frac{\partial u_0}{\partial t} t_0' + \frac{\partial u_0}{\partial x} x_0' \tag{6}$$

Using the initial equation (2), we have a pair of equations for the derivatives $\partial u/\partial t$ and $\partial u/\partial x$, of the form:

$$\begin{aligned} a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} &= c \\ t_0'\frac{\partial u}{\partial t} + x_0'\frac{\partial u}{\partial x} &= u_0' \end{aligned}$$

Basic Linear algebra says that a unique solution exists for these equations exist if and only if

$$\begin{vmatrix} a & b \\ t_0' & x_0' \end{vmatrix} = ax_0' - bt_0' \neq 0$$

If this is not the case then the values of the derivatives become multi-valued[1].

Example 1

Consider the problem:

$$t\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = (x+y)u \quad \text{with} \quad u = 1 \;\; \text{on} \;\; t = 1, x \in (1,2)$$

[1]These are called *shocks* in hydrodynamics

The initial data (at t=1) can be written as:

$$t_0(v) = 1, \quad x_0(v) = v, \quad u_0 = 1, \quad v \in (1, 2)$$

The characteristic equations are:

$$\dot{t} = t, \quad \dot{x} = x, \quad \dot{u} = (x + y)u$$

Solving the first two equations gives:

$$\log x - \log x_0 = s, \quad \log y - \log y_0 = s$$

Now, the initial conditions were given at $s = 1$, so looking back at the Cauchy data gives $t_0 = 1, x_0 = v$ and hence the solutions are:

$$t = e^s, \quad x = ve^s$$

This can be used for the third characteristic equation:

$$\frac{du}{ds} = (x + y)u = e^s(1 + v)u$$

Giving as a solution:

$$\log u - \log u_0(v) = (e^s - 1)(1 + v)$$

As $u_0(v) = 1$ this gives $\log u_0(v) = 0$. The solution is given by:

$$\log u = e^s(1 + v) - (1 + v)$$

Using the equations for t and x the variables s and v can be eliminated to give:

$$\log u = t + x - 1 - \frac{x}{t}$$

Example 2
consider the partial differential equation:

$$x\frac{\partial u}{\partial t} - 2tx\frac{\partial u}{\partial x} = 2tu, \quad \text{with} \;\; u = x^3 \;\; \text{on} \;\; t = 0, x \in [1, 2]$$

The first step is to write out the Cauchy data:

$$x_0(v) = v, \quad t_0(v) = 0, \quad u_0(v) = v^3$$

The characteristic equations are:

$$\dot{t} = x, \quad \dot{x} = -2tx \quad \dot{u} = 2tu$$

As in the previous example the way forward is to take quotients of the characteristic equations, in doing this:

$$\frac{dx}{dt} = \frac{\dot{x}}{\dot{t}} = -2t, \quad \frac{du}{dx} = \frac{\dot{u}}{\dot{x}} = -\frac{u}{x}$$

Integrating these equations give:

$$x + t^2 = x_0 + t_0^2, \quad ux = u_0 x_0$$

Upon using the Cauchy data, these equations become:

$$x + t^2 = v, \quad xu = v^4$$

Eliminating v gives the solution:

$$u = \frac{(x + t^2)^4}{x}$$

The method works for quasilinear first order partial differential equations of the form:

$$a(t, x, u)\frac{\partial u}{\partial t} + b(t, x, u)\frac{\partial u}{\partial x} = c(t, x, u) \tag{7}$$

The following example shows how.
Example 3

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = u^2 \quad \text{with} \ \ u = e^{-x^2} \ \ \text{on} \ \ t = 0$$

The equations for the characteristics are:

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x} &= 1 \\ \dot{u} &= u^2 \end{aligned}$$

The cauchy data is given by:

$$t = 0, \quad x = v, \quad u = e^{-v^2}$$

Solving the first two characteristic equations gives:

$$\begin{aligned} t &= s + t_0(v) \\ x &= s + x_0(v) \end{aligned}$$

Upon using the Cauchy data, the characteristics become:

$$\begin{aligned} t &= s \\ x &= s + v \end{aligned}$$

Eliminating s from the equations shows that $x - t = v$. The third equation is solved to give:

$$-\frac{1}{u} + \frac{1}{u_0(v)} = s$$

Upon using the Cauchy data for u, gives:

$$-\frac{1}{u} + e^{v^2} = s$$

Plugging in the values for s and v yields:

$$u = \frac{1}{e^{(x-t)^2} - t}$$

Note that on the curve $t = \exp(x - t)^2$ the solution does not exist, this means that the domain which is the solution is defined must be restricted to $t < \exp(x - t)^2$. So the method even works for quasilinear equations.

2 Method of Characteristics for Systems of PDEs

This section details the extension of the method of characteristics for systems of equations and will be most useful when applied to the inviscid flow equations where a whole new method of problem solving will open up. For now consider the system of equations as follows:

$$\begin{aligned} a_{11}\frac{\partial u}{\partial t} + a_{12}\frac{\partial v}{\partial t} + a_{13}\frac{\partial w}{\partial t} + b_{11}\frac{\partial u}{\partial x} + b_{12}\frac{\partial v}{\partial x} + b_{13}\frac{\partial w}{\partial x} &= c_1 \\ a_{21}\frac{\partial u}{\partial t} + a_{22}\frac{\partial v}{\partial t} + a_{23}\frac{\partial w}{\partial t} + b_{21}\frac{\partial u}{\partial x} + b_{22}\frac{\partial v}{\partial x} + b_{23}\frac{\partial w}{\partial x} &= c_2 \\ a_{31}\frac{\partial u}{\partial t} + a_{32}\frac{\partial v}{\partial t} + a_{33}\frac{\partial w}{\partial t} + b_{31}\frac{\partial u}{\partial x} + b_{32}\frac{\partial v}{\partial x} + b_{33}\frac{\partial w}{\partial x} &= c_3 \end{aligned}$$

Where the coefficients are all functions of t, x, u, v, w. The system can be written in Matrix form as:

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{c}$$

Where $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}), \mathbf{c} = (c_i)$ and $\mathbf{u} = (u, v, w)$. A small restriction is made upon $\mathbf{A}$ is that $\det \mathbf{A} \neq 0$.

2.1 Cauchy Data and Systems of PDEs

For the purpose of this section the matrices $\mathbf{A}$ and $\mathbf{B}$ are 2×2 but this won't make much difference to the theory as the same holds for systems of more equations. To start with a solution of a 2×2 system

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{c} \tag{8}$$

As before the geometrical interpretation sought after is the description of two surfaces $u = f_1(t, x)$ and $v = f_2(t, x)$. A condition that is expected to hold is that the surfaces representing the solution should pass through the initial curve. this is a boundary condition for the solution surface, such a boundary condition can be written in the form:

$$\mathbf{u} = \mathbf{u}_0(s), \quad x = t_0(s), \quad x = x_0(s) \quad \text{for} \quad s_1 \leq s \leq s_2$$

Differentiating the boundary condition along the initial curve yields:

$$\mathbf{u}_0 = t_0' \frac{\partial \mathbf{u}}{\partial t} + x_0' \frac{\partial \mathbf{u}}{\partial x} \tag{9}$$

The partial derivatives are uniquely defined if and only if

$$\begin{vmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ t_0' & 0 & x_0' & 0 \\ 0 & t_0' & 0 & x_0' \end{vmatrix} \neq 0 \tag{10}$$

denoting $\lambda = x_0'/t_0'$, then adding $-1/\lambda$ lots of the third column to the first column and adding $-1/\lambda$ lots of the fourth column to the second column gives:

$$\begin{vmatrix} a_{11} - \frac{1}{\lambda}b_{11} & a_{12} - \frac{1}{\lambda}b_{12} & b_{11} & b_{12} \\ a_{21} - \frac{1}{\lambda}b_{21} & a_{22} - \frac{1}{\lambda}b_{22} & b_{21} & b_{22} \\ 0 & 0 & x_0' & 0 \\ 0 & 0 & 0 & x_0' \end{vmatrix} = x_0'^2 \begin{vmatrix} a_{11} - \frac{1}{\lambda}b_{11} & a_{12} - \frac{1}{\lambda}b_{12} \\ a_{21} - \frac{1}{\lambda}b_{21} & a_{22} - \frac{1}{\lambda}b_{22} \end{vmatrix} = x_0'^2 \lambda^2 \det(\mathbf{A} - \lambda \mathbf{B}) \neq 0$$

A *characteristic* is a curve in the (t, x) plane defined by:

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0 \tag{11}$$

This means that the characteristics are the curves of discontinuity in the (t, x) plane. As am example of calculating the characteristics, consider the 2D steady linearised gas dynamics equations:

$$\begin{aligned}
\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + U\frac{\partial \rho}{\partial x} &= 0 \\
U\frac{\partial u}{\partial x} + \frac{a_0^2}{\rho_0}\frac{\partial \rho}{\partial x} &= 0 \\
U\frac{\partial v}{\partial x} + \frac{a_0^2}{\rho_0}\frac{\partial \rho}{\partial y} &= 0
\end{aligned}$$

The $\mathbf{A}$ and $\mathbf{B}$ for this system are:

$$\mathbf{A} = \begin{pmatrix} \rho_0 & 0 & U \\ U & 0 & \frac{a_0^2}{\rho_0} \\ 0 & U & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & \rho_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_0^2}{\rho_0} \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} u \\ v \\ \rho \end{pmatrix}$$

Solving the equation $\det(\mathbf{A} - \lambda\mathbf{B}) = 0$ gives:

$$\begin{vmatrix} \rho_0 & -\lambda\rho_0 & U \\ U & 0 & \frac{a_0^2}{\rho_0} \\ 0 & U & -\lambda\frac{a_0^2}{\rho_0} \end{vmatrix} = 0$$

Which gives $\det(\mathbf{A} - \lambda\mathbf{B}) = \lambda a_0^2 U(\lambda^2 + a_0^2 - U^2) = 0$, the characteristics are real if and only if $U^2 > a_0^2$ which implies the linearised flow is supersonic. The characteristics in this case are the streamlines of the unperturbed flow $dx/dt = 0$ and the "Mach lines" $dx/dt = \pm\sqrt{U^2 - a_0^2}$.

2.2 Riemann Invariants

There is still more which can be said and it gives rise to the important concept of a *Riemann Invariant*. Differentiating $\mathbf{u}$ along a characteristic gives:

$$\dot{\mathbf{u}} = \dot{t}\frac{\partial \mathbf{u}}{\partial t} + \dot{x}\frac{\partial \mathbf{u}}{\partial x} \tag{12}$$

Pre-multiply the original system of PDEs by $\dot{t}\mathbf{A}^{-1}$ to give:

$$
\begin{aligned}
\dot{t}\mathbf{A}^{-1}\left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial x}\right) &= \dot{t}\mathbf{A}^{-1}\mathbf{c} \\
\dot{t}\frac{\partial \mathbf{u}}{\partial t} + \dot{t}\mathbf{A}^{-1}\mathbf{B}\frac{\partial \mathbf{u}}{\partial x} &= \dot{t}\mathbf{A}^{-1}\mathbf{c}
\end{aligned}
$$

Substituting for $\dot{t}\partial \mathbf{u}/\partial t$ gives:

$$
\dot{\mathbf{u}} - \dot{x}\mathbf{I}\frac{\partial \mathbf{u}}{\partial x} + \dot{t}\mathbf{A}^{-1}\mathbf{B}\frac{\partial \mathbf{u}}{\partial x} = \dot{t}\mathbf{A}\mathbf{c}
$$

Re-arranging this gives:

$$
\mathbf{A}^{-1}\mathbf{c}\dot{t} - \dot{\mathbf{u}} = (\dot{t}\mathbf{A}^{-1}\mathbf{B} - \dot{x}\mathbf{I})\frac{\partial \mathbf{u}}{\partial x} \tag{13}
$$

Now multiply (13) on the left by an arbitrary row vector ℓ^T to give:

$$
\ell^T(\mathbf{A}^{-1}\mathbf{c}\dot{t} - \dot{\mathbf{u}}) = \ell^T(\dot{t}\mathbf{A}^{-1}\mathbf{B} - \dot{x}\mathbf{I})\frac{\partial \mathbf{u}}{\partial x}
$$

Now if the row vector ℓ^T is chosen in such a way that:

$$
\ell^T(\dot{t}\mathbf{A}^{-1}\mathbf{B} - \dot{x}\mathbf{I}) = 0 \tag{14}
$$

Then the following holds:

$$
\ell^T\mathbf{A}^{-1}\mathbf{c}\dot{t} = \ell^T\dot{\mathbf{u}} \tag{15}
$$

Integrating (15) gives functions which are constant on the characteristics, these functions are called *Riemann Invariants*. The value of λ in this case is given by $\dot{x}/\dot{t} = dx/dt$.

2.3 Examples of Riemann Invariants

Consider the system:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} &= 0 \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} &= 0 \\
\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} &= 0
\end{aligned}
$$

The relevant vector variable here is $\mathbf{u} = (u, v, w)$, the system can be written in matrix notation as:

$$\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{0}$$

Where:

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

Computing $\det(\lambda \mathbf{I} - \mathbf{B}) = 0$ gives:

$$\begin{vmatrix} \lambda - 1 & -1 & 0 \\ -1 & \lambda - 2 & -1 \\ 1 & -2 & -\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

The solutions are $\lambda = \pm 1, 3$. To compute the Riemann invariants the row vector ℓ^T needs to be calculated. Writing $\ell^T = (\alpha, \beta, \gamma)$ the expression (14) becomes:

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} \lambda - 1 & -1 & 0 \\ -1 & \lambda - 2 & -1 \\ 1 & -2 & \lambda \end{pmatrix} = 0$$

This gives three equations:

$$\begin{aligned} (\lambda - 1)\alpha - \beta + \gamma &= 0 \\ -\alpha + (\lambda - 2)\beta - 2\gamma &= 0 \\ -\beta - \lambda\gamma &= 0 \end{aligned}$$

The solution of these equations are:

$$\begin{aligned} \ell^T &= (-3, 1, 1), & \lambda &= 1 \\ \ell^T &= (-1, 1, -1), & \lambda &= -1 \\ \ell^T &= (-2, 3, 1), & \lambda &= 3 \end{aligned}$$

Then the Riemann invariants are:

$$\begin{aligned} R_1 &= -3u + v + w, & dy/dx &= 1 \\ R_2 &= -u + v - w, & dy/dx &= -1 \\ R_3 &= -2u + 3v + w, & dy/dx &= 3 \end{aligned}$$

The next example is the unsteady Euler equations in the form:

$$\begin{aligned}\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) &= 0\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x} &= a^2\left[\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right]\end{aligned}$$

This can be written in the form:

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{u}}{\partial x}=\mathbf{0} \tag{16}$$

Where $\mathbf{u}=(\rho,u,p)$ and:

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -a^2 & 0 & 1\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ -ua^2 & 0 & u\end{pmatrix}$$

Multiplying throughout by $\mathbf{A}^{-1}$ gives and equation of the form:

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{C}\frac{\partial \mathbf{u}}{\partial x}=\mathbf{0}$$

where

$$\mathbf{C}=\begin{pmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & a^2\rho & u\end{pmatrix}$$

The next task is to calculate the eigenvalues of $\mathbf{C}$, this is a straightforward calculation, yielding three separate eigenvalues: $\lambda=u,u+a,u-a$. To calculate the left eigenvectors, compute:

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}u-\lambda & \rho & 0\\ 0 & u-\lambda & 1/\rho\\ 0 & a^2\rho & u-\lambda\end{pmatrix}=\mathbf{0}$$

This reduces to three linear equations:

$$\begin{aligned}\alpha(u-\lambda) &= 0\\ \alpha\rho+(u-\lambda)\beta+a^2\rho\gamma &= 0\\ \beta/\rho+\gamma(u-\lambda) &= 0\end{aligned}$$

Looking at the eigenvalues $\lambda=u\pm a$ shows that $\alpha=0$ and $\beta=\pm\rho a\gamma$. Taking $\beta=1$ and $\gamma=\pm 1/\rho a$ gives the corresponding eigenvectors to be

$\ell^T = (0, 1, \pm 1/\rho a)$. The derivatives of the Riemann invariants are then given by:

$$dR_+ = du + \frac{dp}{\rho a} \quad \text{on} \quad \frac{dx}{dt} = u + a \tag{17}$$

$$dR_- = du - \frac{dp}{\rho a} \quad \text{on} \quad \frac{dx}{dt} = u - a \tag{18}$$

The Lagrangian equations of fluid mechanics can also be cast into characteristic form. The equations in Lagrangian form are:

$$\frac{\partial}{\partial t}\left(\frac{1}{\nu}\frac{\partial x}{\partial a}\right) = 0 \tag{19}$$

$$\frac{\partial u}{\partial t} + \nu_0 \frac{\partial p}{\partial a} = 0 \tag{20}$$

$$a^2 = -\nu^2 \frac{\partial p}{\partial \nu} \tag{21}$$

Expanding (19) gives:

$$\frac{\partial \nu}{\partial t} - \nu_0 \frac{\partial u}{\partial a} = 0 \tag{22}$$

Equation (21) can be written as:

$$\frac{\partial \nu}{\partial t} + \frac{\nu^2}{a^2}\frac{\partial p}{\partial t} = 0 \tag{23}$$

Equations (22), (20) and (23) make up the equations required to carry out a characteristic analysis. Use:

$$\begin{aligned} \frac{\partial \nu}{\partial t} - \nu_0 \frac{\partial u}{\partial a} &= 0 \\ \frac{\partial u}{\partial t} + \nu_0 \frac{\partial p}{\partial a} &= 0 \\ \frac{\partial \nu}{\partial t} + \frac{\nu^2}{a^2}\frac{\partial p}{\partial t} &= 0 \end{aligned}$$

This can be written in the form:

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \tag{24}$$

Where $\mathbf{u} = (\nu, u, p)$ and:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & \nu^2/a^2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -\nu_0 & 0 \\ 0 & 0 & \nu_0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiplying throughout by $\mathbf{A}^{-1}$ gives and equation of the form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{C}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}$$

where

$$\mathbf{C} = \begin{pmatrix} 0 & -\nu_0 & 0 \\ 0 & 0 & \nu_0 \\ 0 & C^2/\nu_0 & 0 \end{pmatrix}$$

The eigenvalues (λ) of $\mathbf{C}$ are given by $\det(\mathbf{C} - \lambda\mathbf{I}) = 0$. The eigenvalues are then given by $\lambda = 0, \pm C$. The next task is to compute the eigenvectors of $\mathbf{C}$, they are given by:

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} -\lambda & -\nu_0 & 0 \\ 0 & -\lambda & \nu_0 \\ 0 & C^2/\nu_0 & -\lambda \end{pmatrix} = \mathbf{0}$$

For $\lambda \neq 0$, $\alpha = 0$ and the three equations reduce to $\beta = \rho_0 \lambda \gamma$. A solution of this equation is $\beta = 1$ and $\gamma = \pm\rho_0 C$ for $\lambda = \pm C$. The Riemann invariants ($R_\pm$) on the characteristics $da/dt = \pm C$ are:

$$R_+ = u + \int \frac{dp}{\rho_0 C} \quad \text{on} \quad \frac{da}{dt} = C \tag{25}$$

$$R_- = u - \int \frac{dp}{\rho_0 C} \quad \text{on} \quad \frac{da}{dt} = -C \tag{26}$$

On their respective characteristics the Riemann invariants are constant, so taking the positive characteristic om the free surface and noting that $dp = 0$ on the free surface, the Riemann invariant is $R_+ = u_{fs}$. The same Riemann invariant in the material will give the particle velocity and so:

$$u_{fs} = \frac{1}{2}\left(u + \int \frac{dp}{\rho_0 C}\right) \tag{27}$$

which is valid on $da/dt = +C$.