Circle Inversion Question (2 Geometric)

Suppose that C is the unit circle around the origin. Let $(x-a)^2+(y-b)^2 = c^2$ be any other circle D not passing through the origin. Prove analytically that the inversion of the circle D with respect to C will still be a circle.

I decided to try a very different approach. A familiar theorem, which I've seen proved synthetically, tells us what happens when one circle is inverted with respect to another. Except in two special cases (when the circle being inverted passes through the center of inversion, and when the circle being inverted is orthogonal to the circle of inversion; but these cases are not of relevancy here) the image (1) will be a third circle; (2) the centers of the three circles will be collinear; and (3) the radius of the image circle, which we can call D', will be

$$c\frac{k^2}{p} \tag{1}$$

where c is the radius of the circle we call D (the one being inverted), k is the radius of the circle of inversion, and p is the power of the center of inversion with respect to D. In this problem, we're assuming that the circle of inversion is a unit circle, so the radius of D' is just

$$\frac{c}{p} \tag{2}$$

I figured that if I illustrated this on a coordinate system, as on page 1 of the GSP file, then I could derive a formula for the image circle D'.

On page 2, I show a different view of D, the circle being inverted. It's center is at (a, b), so its distance from the origin is $\sqrt{a^2+b^2}$. along the line

$$y = \frac{b}{a}x \tag{3}$$

The two points u and v are the points where this line intersects D, so they are the opposite points of a diameter. Their distances from the origin, which I've labeled O, are therefore

$$\begin{aligned} \|Ou\| &= \sqrt{a^2+b^2} - c && (4) \\ \|Ov\| &= \sqrt{a^2+b^2} + c && (5) \end{aligned}$$

and the power of O with respect to D is

$$\|Ou\|\,\|Ov\| = a^2+b^2-c^2 \tag{6}$$

Using the formula on line 2, I can already calculate the radius of the image circle D' to be

$$\frac{c}{a^2+b^2-c^2} \tag{7}$$

Now all I need is the center of the image circle. My plan is to find the coordinates of u and v, calculate their images u' and v', and find the midpoint. I can also find the distance between them $\|u'v'\|$, which should be twice the radius of D', so I can check the calculation above.

So how do I find the coordinates of u and v? On page 3 of the GSP file, in effect I translated the circle D to the origin, and found a point (x, y) where it intersects the line $y = \frac{b}{a}x$. This means

$$\begin{aligned}
c &= \sqrt{x^2 + y^2} && (8) \\
c &= \sqrt{x^2 + \frac{b^2}{a^2}x^2} && (9) \\
c &= |x|\sqrt{1 + \frac{b^2}{a^2}} && (10) \\
c &= |x|\sqrt{\frac{a^2 + b^2}{a^2}} && (11) \\
|x| &= \frac{ca}{\sqrt{a^2 + b^2}} && (12) \\
x &= \frac{ca}{\sqrt{a^2 + b^2}} && (13)
\end{aligned}$$

I eliminate the absolute value in the last line because $x < 0$ if and only if $a < 0$. In a similar way, I find that

$$y = \frac{cb}{\sqrt{a^2 + b^2}} \tag{14}$$

The coordinates of the point u are

$$\begin{aligned}
u &= \begin{bmatrix} a - \left(\frac{ac}{\sqrt{a^2+b^2}}\right) \\ \left(\frac{b}{a}\right)\left(a - \left(\frac{ac}{\sqrt{a^2+b^2}}\right)\right) \end{bmatrix} && (15) \\
u &= \begin{bmatrix} -a\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \\ -b\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \end{bmatrix} && (16)
\end{aligned}$$

and the coordinates of the point v are, similarly,

$$v = \begin{bmatrix} a\frac{c+\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \\ b\frac{c+\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} \end{bmatrix} \tag{17}$$

Note: I made use of the fact that u and v lie on a line, the slope of which I know, so their values are proportional. Now that I know these coordinates, I can use the conversion formulas under circle inversion in a unit circle, which are

$$\begin{aligned}
x' &= \frac{x}{x^2 + y^2} && (18) \\
y' &= \frac{y}{x^2 + y^2} && (19)
\end{aligned}$$

to find the coordinates of u' and v'.

$$x' = \left[\frac{x}{x^2+y^2}\right]_{x=-a\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}},\, y=-b\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}} \quad (20)$$

$$x' = -a\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}\left(a^2+b^2-c^2\right)^2}\left(2c\sqrt{a^2+b^2}+a^2+b^2+c^2\right) \quad (21)$$

$$y' = \left[\frac{y}{x^2+y^2}\right]_{x=-a\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}},\, y=-b\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}} \quad (22)$$

$$y' = -b\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}\left(a^2+b^2-c^2\right)^2}\left(2c\sqrt{a^2+b^2}+a^2+b^2+c^2\right) \quad (23)$$

Therefore, the coordinates of u' are

$$u' = \begin{bmatrix} -a\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}(a^2+b^2-c^2)^2}\left(2c\sqrt{a^2+b^2}+a^2+b^2+c^2\right), \\ -b\frac{c-\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}(a^2+b^2-c^2)^2}\left(2c\sqrt{a^2+b^2}+a^2+b^2+c^2\right) \end{bmatrix} \quad (24)$$

Similarly, the coordinates of $v'$are

$$v' = \begin{bmatrix} a\frac{c+\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}(a^2+b^2-c^2)^2}\left(a^2-2c\sqrt{a^2+b^2}+b^2+c^2\right) \\ b\frac{c+\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}(a^2+b^2-c^2)^2}\left(a^2-2c\sqrt{a^2+b^2}+b^2+c^2\right) \end{bmatrix}$$

Notice that the two points are in similar form; the only difference relates to plus and minus signs. Next I need to find the midpoint of u' and v', that is, the center of D'. I'll omit the calculation, which though simple takes up much space on a page. The result is:the point

$$\begin{bmatrix} \frac{a}{a^2+b^2-c^2} \\ \frac{b}{a^2+b^2-c^2} \end{bmatrix} \quad (25)$$

And what's the radius of this image circle? Again, the calculation looks messy on the page, though simple in concept. The result is

$$\frac{c}{a^2+b^2-c^2} \quad (26)$$

Just as expected. So the formula for the image circle seems to be

$$\left(x'-\frac{a}{a^2+b^2-c^2}\right)^2+\left(y'-\frac{b}{a^2+b^2-c^2}\right)^2=\frac{c^2}{\left(a^2+b^2-c^2\right)^2} \quad (27)$$

I made several graphs of circles D and D' from this formula, using various values of a, b and c, and they all look reasonable. I'm satisfied that my formula is correct. But I'm still unable to derive it algebraically.

Remember in the previous file, I derived the following formula for D':

$$\left(x' - a\frac{x}{x'}\right)^2 + \left(y' - b\frac{y}{y'}\right)^2 = c^2\frac{xy}{x'y'} \tag{28}$$

This seems to mean that

$$\frac{x}{x'} = \frac{y}{y'} = \frac{1}{a^2 + b^2 - c^2} \tag{29}$$

If I substitute certain values, I get plausible results, but I can't find an argument for it.