

I am reading Maxwell's "a treatise on electricity and magnetism, Volume 2, page 156" about "Ampere's Force Law". I have some confusion in the following pages:

The coordinates of points on either current are functions of  $s$  or of  $s'$ .

If  $F$  is any function of the position of a point, then we shall use the subscript  $(s,0)$  to denote the excess of its value at  $P$  over that at  $A$ , thus

$$F_{(s,0)} = F_P - F_A.$$

Such functions necessarily disappear when the circuit is closed.

Let the components of the total force with which  $A'P$  acts on  $AA$  be  $ii'X$ ,  $ii'Y$ , and  $ii'Z$ . Then the component parallel to  $X$  of the force with which  $ds'$  acts on  $ds$  will be  $ii' \frac{d^2X}{ds ds'} ds ds'$ .

$$\text{Hence} \quad \frac{d^2X}{ds ds'} = R \frac{\xi}{r} + Sl + S'l'. \quad (13)$$

Substituting the values of  $R$ ,  $S$ , and  $S'$  from (12), remembering that

$$l\xi + m'\eta + n'\zeta = r \frac{dr}{ds'}, \quad (14)$$

and arranging the terms with respect to  $l, m, n$ , we find

$$\begin{aligned} \frac{d^2X}{ds ds'} &= l \left\{ -(A+B) \frac{1}{r^2} \frac{dr}{ds'} \xi^2 + C \frac{dr}{ds'} + (B+C) \frac{l'\xi}{r} \right\}, \\ &+ m \left\{ -(A+B) \frac{1}{r^2} \frac{dr}{ds'} \xi \eta + C \frac{l'\eta}{r} + B \frac{m'\xi}{r} \right\}, \\ &+ n \left\{ -(A+B) \frac{1}{r^2} \frac{dr}{ds'} \xi \zeta + C \frac{l'\zeta}{r} + B \frac{n'\xi}{r} \right\}. \end{aligned} \quad (15)$$

Since  $A$ ,  $B$ , and  $C$  are functions of  $r$ , we may write

$$P = \int_r^\infty (A+B) \frac{1}{r^2} dr, \quad Q = \int_r^\infty C dr, \quad (16)$$

the integration being taken between  $r$  and  $\infty$  because  $A$ ,  $B$ ,  $C$  vanish when  $r = \infty$ .

$$\text{Hence} \quad (A+B) \frac{1}{r^2} = -\frac{dP}{dr}, \quad \text{and} \quad C = -\frac{dQ}{dr}. \quad (17)$$

516.] Now we know, by Ampère's third case of equilibrium, that when  $s'$  is a closed circuit, the force acting on  $ds$  is perpendicular to the direction of  $ds$ , or, in other words, the component of the force in the direction of  $ds$  itself is zero. Let us therefore assume the direction of the axis of  $x$  so as to be parallel to  $ds$  by making  $l = 1$ ,  $m = 0$ ,  $n = 0$ . Equation (15) then becomes

$$\frac{d^2X}{ds ds'} = \frac{dP}{ds'} \xi^2 - \frac{dQ}{ds'} + (B+C) \frac{l'\xi}{r}. \quad (18)$$

To find  $\frac{d^2X}{ds}$ , the force on  $ds$  referred to unit of length, we must

integrate this expression with respect to  $s'$ . Integrating the first term by parts, we find

$$\frac{dX}{ds} = (P\xi^2 - Q)_{(s,0)} - \int_0^{s'} (2Pr - B - C) \frac{\nu'\xi}{r} ds'. \quad (19)$$

When  $s'$  is a closed circuit this expression must be zero. The first term will disappear of itself. The second term, however, will not in general disappear in the case of a closed circuit unless the quantity under the sign of integration is always zero. Hence, to satisfy Ampère's condition,

$$P = \frac{1}{2r}(B + C). \quad (20)$$

517.] We can now eliminate  $P$ , and find the general value of

$$\begin{aligned} \frac{dX}{ds}, \quad \frac{dX}{ds} = & \left\{ \frac{B+C}{2} \frac{\xi}{r} (l\xi + m\eta + n\zeta) + Q \right\}_{(s,0)} \\ & + m \int_0^{s'} \frac{B-C}{2} \frac{m'\xi - l'\eta}{r} ds' - n \int_0^{s'} \frac{B-C}{2} \frac{l'\zeta - n'\xi}{r} ds'. \end{aligned} \quad (21)$$

When  $s'$  is a closed circuit the first term of this expression vanishes, and if we make

$$\left. \begin{aligned} \alpha' &= \int_0^{s'} \frac{B-C}{2} \frac{n'\eta - m'\zeta}{r} ds', \\ \beta' &= \int_0^{s'} \frac{B-C}{2} \frac{l'\zeta - n'\xi}{r} ds', \\ \gamma' &= \int_0^{s'} \frac{B-C}{2} \frac{m'\xi - l'\eta}{r} ds', \end{aligned} \right\} \quad (22)$$

where the integration is extended round the closed circuit  $s'$ , we may write

$$\left. \begin{aligned} \frac{dX}{ds} &= m\gamma' - n\beta', \\ \text{Similarly} \quad \frac{dY}{ds} &= n\alpha' - l\gamma', \\ \frac{dZ}{ds} &= l\beta' - m\alpha'. \end{aligned} \right\} \quad (23)$$

The quantities  $\alpha'$ ,  $\beta'$ ,  $\gamma'$  are sometimes called the determinants of the circuit  $s'$  referred to the point  $P$ . Their resultant is called by Ampère the directrix of the electrodynamic action.

It is evident from the equation, that the force whose components are  $\frac{dX}{ds}$ ,  $\frac{dY}{ds}$ , and  $\frac{dZ}{ds}$  is perpendicular both to  $ds$  and to this directrix, and is represented numerically by the area of the parallelogram whose sides are  $ds$  and the directrix.

**My question is of two parts:**

1. Equation 20, i.e.  $P = \frac{B+C}{2r}$  is the outcome of special case (i.e.  $l=1$ ,  $m=0$ ,  $n=0$ )

But in Page 156, Article 517, Maxwell says: "We can now eliminate P, and find the general value of  $\frac{dX}{ds}$ " and uses this formula (i.e.  $P = \frac{B+C}{2r}$ ) in the general case.

However in the general case, where  $0 < l, m, n < 1$ , and hence

$$\frac{d^2X}{dsds'} = l \left( \frac{dP}{ds'} \xi^2 - \frac{dQ}{ds'} + (B+C) \frac{l'\xi}{r} \right) + m(\dots) + n(\dots) \neq 0$$

(since direction of X is not in the direction of ds)  
therefore,

$$\frac{dX}{ds} = l \left[ (P\xi^2 - Q)_{(s',0)} - \int_0^{s'} (2Pr - B - C) \frac{l'\xi}{r} ds' \right] + m \int_0^{s'} (\dots) ds' + n \int_0^{s'} (\dots) ds'$$

Now in this general case, how can we get  $P = \frac{B+C}{2r}$ .

If  $P \neq \frac{B+C}{2r}$  in general case, what does Maxwell mean by "We can now eliminate P, and find the general value of  $\frac{dX}{ds}$ ,"

2. How can one get equation 21 from equation 15. Please give a lengthy derivation.