Free Surface Flows in Electrohydrodynamics with Constant Charge

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Abstract

Research on free-surface flows in electrohydrodynamics has predominantly focused on the limiting cases of perfectly conducting and perfectly insulating fluids. In the former, the electric field vanishes within the fluid, whereas in the latter, the electric field persists, but no free charge is present. For perfectly conducting fluids, the coupling between the electric field and the fluid arises solely through the boundary conditions. In the present analysis, we examine the intermediate case in which the charge density within the fluid is uniform. This assumption gives rise to a static electric field that serves as a forcing mechanism for the fluid velocity. On this basis, we derive a global Bernoulli equation that includes an additional contribution dependent on the potential difference within the fluid.

1 Introduction and Set Up

Much of the previous work on free surface flows has been concerned with either infinitely conducting or infinitely insulating fluids, which result in no charge within the fluid. In certain circumstances, this is a reasonable modelling assumption; however, it may not apply to all fluids. In this analysis, the assumption that there is no charge in the fluid is relaxed, but is constant. The assumption of an irrotational and incompressible fluid is still kept, as well as the electric field above the fluid being charge-free. The set up is demonstrated in figure 1.

In region 2, we consider an irrotational and incompressible fluid. The velocity field of the fluid is given by $\mathbf{u}_2$, and the requirement of irrotationality is given by $\nabla \times \mathbf{u}_2 = 0$. As $\nabla \times \mathbf{u}_2 = 0$, there exists a velocity potential function defined by $\mathbf{u}_2 = \nabla\varphi_2$. The requirement of incompressibility shows that:

$$\nabla^2\varphi_2 = 0 \tag{1}$$

Let the constant charge density be q, and as there are no magnetic fields, this demonstrates that the electric field, $\mathbf{E}$, is irrotational, showing that $\nabla \times \mathbf{E}_2 = \mathbf{0}$. so there is a potential difference, V_2 defined by $\mathbf{E}_2 = \nabla V_2$. Inserting this into

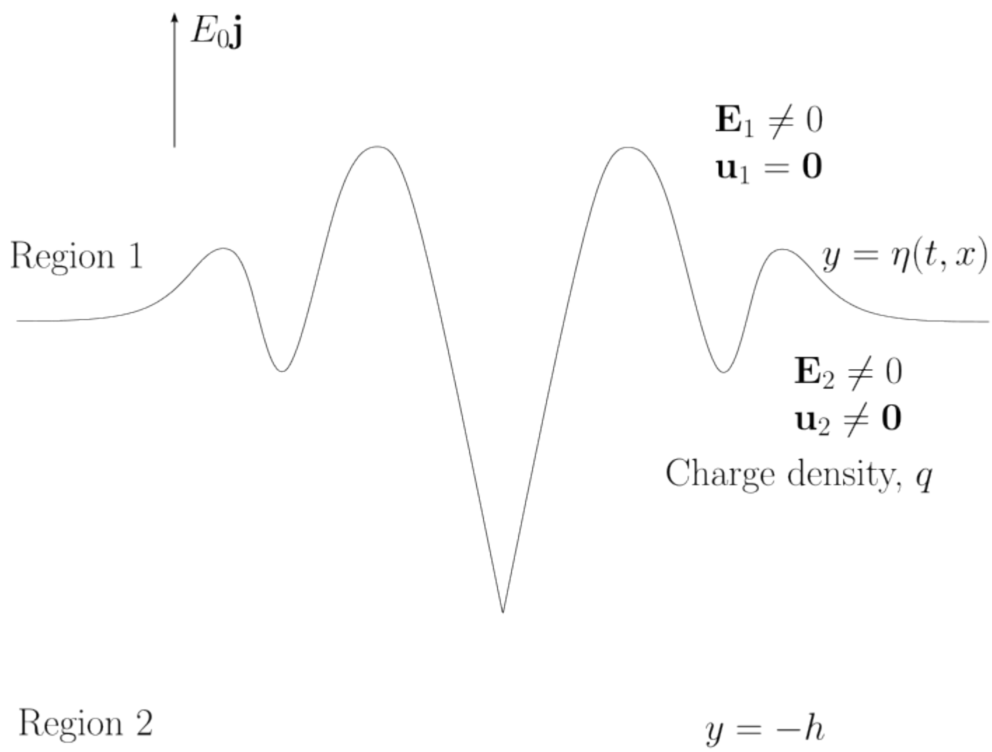

Figure 1: Set up of problem

Gauss's law of electromagnetism shows that:

$$\nabla^2 V_2 = \frac{q}{\epsilon} \tag{2}$$

To include the electric field within the equations of motion, the Maxwell tensor is used:

$$\Sigma_{ij} = \epsilon_p \left(\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} - \frac{1}{2} \delta_{ij} \sum_{k=1}^{2} \frac{\partial V}{\partial x_k} \frac{\partial V}{\partial x_k} \right). \tag{3}$$

Where ϵ_p is called the electric permittivity, the tensor Σ_{ij} has various names; in the fluids literature, it is called the Maxwell-stress tensor. The individual components of the Maxwell tensor are:

$$\begin{aligned}
\Sigma_{11} &= \frac{\epsilon_p}{2} \left[\left(\frac{\partial V}{\partial x} \right)^2 - \left(\frac{\partial V}{\partial y} \right)^2 \right] \\
\Sigma_{12} &= \epsilon_p \frac{\partial V}{\partial x} \frac{\partial V}{\partial y} \\
\Sigma_{22} &= -\frac{\epsilon_p}{2} \left[\left(\frac{\partial V}{\partial x} \right)^2 - \left(\frac{\partial V}{\partial y} \right)^2 \right]
\end{aligned}$$

The Navier-Stokes equations are:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \nabla \cdot \mathbf{T} - g\mathbf{j} \tag{4}$$

The stress tensor is the sum of the pressure and Maxwell stresses:

$$T_{ij} = -p\delta_{ij} + \Sigma_{ij} \tag{5}$$

The divergence of the stress tensor is given by:

$$\begin{aligned}
(\nabla \cdot \mathbf{T})_j &= \partial_i T_{ij} \\
&= -\delta_{ij} \partial_i p + \partial_i \Sigma_{ij} \\
&= -\partial_j p + \epsilon_p \left(\partial_i \left(E_i E_j \right) - \frac{1}{2} \delta_{ij} \partial_i (E_k E_k) \right) \\
&= -\partial_j p + \epsilon_p \left(\partial_i (\partial_{i}) V \partial_j V) - \frac{1}{2} \partial_j (\partial_k V \partial_k V) \right) \\
&= -\partial_j p + \epsilon_p \left(\partial_i^2 V \partial_j V + \partial_i V \partial_i \partial_j V - \frac{1}{2} \partial_j \left[\partial_k V \right]^2 \right) \\
&= -\partial_j p + q \partial_j V + \epsilon_p \left(\partial_i V \partial_i \partial_j V - \frac{1}{2} \partial_j \left[\partial_k V \right]^2 \right) \\
&= -\partial_j p + q \partial_j V + \epsilon_p \left(\frac{1}{2} \left[\partial_i V \right]^2 - \frac{1}{2} \left[\partial_i V \right]^2 \right) \\
&= -\partial_j p + q \partial_j V
\end{aligned}$$

The Navier-Stokes equations become:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} - \frac{q}{\rho}\nabla V_2 \tag{6}$$

As $\mathbf{u}_2$ is irrotational, the advection term can be written as:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}|\nabla\varphi|^2\right) \tag{7}$$

The Navier-Stokes equation becomes:

$$\nabla\left(\frac{\partial \varphi_2}{\partial t} + \frac{1}{2}|\nabla\varphi_2|^2\right) = \nabla\left(-\frac{p}{\rho}\right) + \nabla\left(\frac{qV_2}{\rho}\right) - \nabla(gy) \tag{8}$$

This can be integrated to get a global Bernoulli equation:

$$\frac{\partial \varphi_2}{\partial t} + \frac{1}{2}|\nabla_2\varphi|^2 + \frac{p_2}{\rho} - \frac{qV_2}{\epsilon} + gy = C \tag{9}$$

Equation (9) is valid in the fluid region. This immediately generalises the previous work, where there is no electric term within the Bernoulli equation. The governing equation for the electric potential in region 1 is given by:

$$\nabla^2 V_1 = 0 \tag{10}$$

This completes the governing equations within the regions. The next part of the set-up is to consider boundary conditions. We are interested in capillary responses in this analysis, and so we include the Young-Laplace equation:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_2^1 = \sigma\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{\frac{3}{2}}} \tag{11}$$

The equation for the free surface is given by the expression $f(x) - y = 0$. The implicit assumption is that the free surface can be represented as a graph. The unit normal is given by:

$$\hat{\mathbf{n}} = \frac{\partial_x\eta\mathbf{i} - \mathbf{j}}{\sqrt{1+(\partial_x\eta)^2}} \tag{12}$$

The Young-Laplace equation reduces to:

$$p_2 = \mathcal{P} + \hat{n}_i\Sigma_{ij}^{(1)}\hat{n}_j - \hat{n}_i\Sigma_{ij}^{(2)}\hat{n}_j - \sigma\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{\frac{3}{2}}} \tag{13}$$

where $\Sigma^{(k)}$ denotes the Maxwell stress tensor in region k. The Bernoulli equation (9) can be evaluated on the free-surface to obtain:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + \frac{p_2}{\rho} - \frac{qV_2}{\rho} + g\eta = C \tag{14}$$

The final Bernoulli equation on the boundary is:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla \varphi|^2 + \frac{\mathcal{P}}{\rho} - \frac{qV_2}{\epsilon_2} + \hat{n}_i \left[\Sigma_{ij}\right]_2^1 \hat{n}_j + g\eta = \frac{\sigma}{\rho} \frac{\partial_x \eta}{(1 + (\partial_x \eta)^2)^{\frac{3}{2}}} + C \tag{15}$$

The interfacial condition for electric fields are:

$$\left[\mathbf{E} \cdot \hat{\mathbf{t}}\right]_1^2 = 0 \tag{16}$$

In terms of potentials, this is:

$$\frac{\partial V_1}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V_1}{\partial y} = \frac{\partial V_2}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial V_2}{\partial y} \tag{17}$$

This however, can be simplified by stating that the potentials are continuous across the interface and so:

$$V_1(t, x, \eta(t, x)) = V_2(t, x, \eta(t, x)) \tag{18}$$

The other boundary condition of interest will be the continuity of electric displacement across the free surface.

$$\epsilon_1 \mathbf{E}_1 \cdot \hat{\mathbf{n}} = \epsilon_2 \mathbf{E}_2 \cdot \hat{\mathbf{n}}, \quad \text{on} \quad y = \eta(t, x) \tag{19}$$

In terms of the electric potential, this is:

$$\epsilon_1 \left(-\frac{\partial \eta}{\partial x}\frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial y} \right) = \epsilon_2 \left(-\frac{\partial \eta}{\partial x}\frac{\partial V_2}{\partial x} + \frac{\partial V_2}{\partial y} \right) \quad \text{on} \quad y = \eta(t, x) \tag{20}$$

At $y = -h$ we require that $E_2 = \mathbf{0}$ and that $\mathbf{E}_1 \to E_0 \mathbf{j}$ as $y \to \infty$.

2 Electrostatic Equilibrium and the Bernoulli Equation

In this section electrostatic equilibrium is considered. The fluid is moving with a uniform velocity U. As the fluid is infinite in horizontal region, so in region 2 we can write Gauss's law as:

$$\frac{d^2 V_2}{dy^2} = \frac{q}{\epsilon_2} \tag{21}$$

Integrating once shows that:

$$\frac{dV_2}{dy} = \frac{qy}{\epsilon_2} + a \tag{22}$$

As the electric field below the fluid is zero, this implies that:

$$\frac{dV_2}{dy} = \frac{q}{\epsilon_2}(y + h) \tag{23}$$

Integrating once more to obtain:

$$V_2 = \frac{q}{2\epsilon_2}(y+h)^2 + b \tag{24}$$

as the electric field in the far field must be $E_0\mathbf{j}$, this implies that:

$$V_1 = E_0 y \tag{25}$$

Equating the electric potentials at $y = 0$ shows that:

$$\epsilon_1 E_0 = \frac{qh^2}{2} + b\epsilon_2 \tag{26}$$

Showing that:

$$b = \bar{\epsilon}E_0 - \frac{qh^2}{2\epsilon_2}, \quad \text{where} \quad \bar{\epsilon} = \epsilon_1/\epsilon_2. \tag{27}$$

Making V_2 to be:

$$V_2 = \frac{q}{2\epsilon_2}(y^2 + 2hy) + \bar{\epsilon}E_0 \tag{28}$$

To check, the tangential components of the electric field match on the interface. As the fluid is moving with bulk velocity U, the velocity potential is $\varphi = Ux$. Using this information, it is now possible to compute the constant C in eq (15). In order to calculate this, $\partial_y V$ needs to be known in both regions on the boundary at $y = 0$. They are:

$$\left.\frac{\partial V_1}{\partial y}\right|_{y=0} = E_0, \quad \left.\frac{\partial V_2}{\partial y}\right|_{y=0} = \frac{qh}{\epsilon_2} \tag{29}$$

The valocity profile that is required is:

$$\frac{\partial \varphi}{\partial y} = U \tag{30}$$

The free surface is given by $\eta = 0$, so the final Bernoulli equation is:

$$U^2 - \frac{\epsilon_1 E_0^2}{2} - \frac{q^2h^2}{2\epsilon_2} - \frac{q\epsilon_1 E_0}{\epsilon_2^2} = C \tag{31}$$