ON LAGRANGIAN SYSYEMS WITH CONSTRAINTS

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Abstract. ....

1. Introduction

2. The Main Theorem

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3. Connections and Distributions

In the sequel we regard all the objects to be smooth.

Let M be a manifold with local coordinates $x = (x^1, \ldots, x^m)$. Assume that M is equipped with a symmetric connection

$$\{\Gamma^k_{ij}(x)\}, \quad \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Recall that if we are given with a curve $\gamma : [0, 1] \to M$ then the parallel transport

$$G^s_{s'}(\gamma) : T_{\gamma(s')}M \to T_{\gamma(s)}M$$

is defined as follows

$$\begin{aligned} &\frac{d}{ds}\big(G^s_{s'}(\gamma)\hat v\big)^r + \Gamma^r_{ij}(\gamma(s))\dot\gamma^i(s)\big(G^s_{s'}(\gamma)\hat v\big)^j = 0,\\ &G^{s'}_{s'}(\gamma)\hat v = \hat v \in T_{\gamma(s')}M. \end{aligned} \tag{3.1}$$

Let

$$\omega^\tau(x) = \omega^\tau_i(x)dx^i, \quad \tau = 1, \ldots, n < m, \quad \mathrm{rang}\,(\omega^\tau_i) = n$$

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be 1-forms that define a distribution

$$p(x)=\bigcap_{\tau=1}^{n}\ker\omega^\tau(x)\subset T_xM$$

on M.

Theorem 1. *Assume that there exist 1-forms*

$$\alpha^\tau_\psi=\alpha^\tau_{\psi i}dx^i,\quad \tau,\psi=1,\ldots,n$$

such that

$$\nabla_q\omega^\tau_\lambda=\alpha^\tau_{\psi q}\omega^\psi_\lambda. \tag{3.2}$$

Then for any curve γ it follows that

$$G^s_{s'}(\gamma)p(\gamma(s'))=p(\gamma(s)).$$

Proof. Introduce a notation $v(s)=G^s_{s'}(\gamma)\hat v,\quad \hat v\in p(\gamma(s'))$.

Multiply both sides of equation (3.2) by $\dot\gamma^q(s)v^\lambda(s)$:

$$v^\lambda(s)\frac{d}{ds}\omega^\tau_\lambda(\gamma(s))-\Gamma^r_{q\lambda}(\gamma(s))\omega^\tau_r(\gamma(s))v^\lambda(s)\dot\gamma^q(s)=\dot\gamma^q(s)v^\lambda(s)\alpha^\tau_{\psi q}\omega^\psi_\lambda. \tag{3.3}$$

Multiply both sides of equation (3.1) by $\omega^\tau_r(\gamma(s))$:

$$\omega^\tau_r\frac{d}{ds}v^r(s)+\omega^\tau_r\Gamma^r_{ij}\dot\gamma^i v^j=0. \tag{3.4}$$

Introduce a notation $y^\tau(s)=\omega^\tau_r(\gamma(s))v^r(s)$. Note that $y^\tau(s')=0$.

Adding (3.4) and (3.3) we get

$$\dot y^\tau=\dot\gamma^q\alpha^\tau_{\psi q}y^\psi.$$

By the Cauchy existence and uniqueness theorem we obtain $y^\tau(s)=0$.

The Theorem is proved.

Theorem 2. *Assume that there exist 1-forms*

$$\alpha^\tau_\psi=\alpha^\tau_{\psi i}dx^i,\quad \beta^\tau_\psi=\beta^\tau_{\psi i}dx^i\quad \tau,\psi=1,\ldots,n$$

such that

$$\nabla_q\omega^\tau_\lambda=\alpha^\tau_{\psi q}\omega^\psi_\lambda+\beta^\tau_{\psi\lambda}\omega^\psi_q. \tag{3.5}$$

Then for any curve γ such that $\dot\gamma(s)\in p(\gamma(s))$ one has

$$G^s_{s'}(\gamma)p(\gamma(s'))=p(\gamma(s)).$$

This theorem is proved in the same manner as the previous one.

References

[1] R. A. Adams J.J.F. Fournier: Sobolev Spaces, Elsevier, 2nd Edn. 2003.