Find the function $f(x)$ where:

$$\mathscr{L}(f(x)) = \frac{1}{(s-a)\sqrt{s^2-b^2}}$$

To compute the inver Laplace transform, the Bromwich contour has to be used with (in this case) a branch cut from $-b$ to b There is a simple pole at $s=a$,

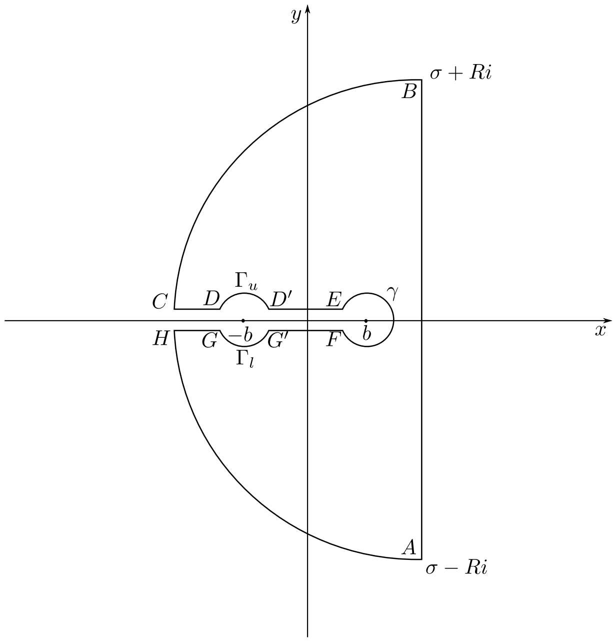

Figure 1: Contour of Integration

the residue of which is $e^{at}/\sqrt{a^2-b^2}$. So:

$$\oint \frac{e^{xs}}{(s-a)\sqrt{s^2-b^2}} ds = 2\pi i \frac{e^{at}}{\sqrt{a^2-b^2}}$$

The contour can be split up into 10 pieces. Examining the BC integral:

$$I_{BC} = \int_{\frac{\pi}{2}}^{\pi} \frac{e^{xRe^{i\theta}}}{(Re^{i\theta}-a)\sqrt{R^2e^{2i\theta}-b^2}}$$

The BC contour is given by $s = Re^{i\theta}$ and so $ds = Rie^{i\theta}d\theta$. Basic inequalities show that:

$$|Re^{i\theta}-a| \geqslant |R-|a||, \quad |\sqrt{R^2e^{2i\theta}-b^2}| \geqslant \sqrt{|R^2-b^2|}$$

Now:

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow xRe^{i\theta} = xR\cos\theta + ixR\sin\theta$$

So:

$$|e^{Rxe^{i\theta}}| = e^{xR\cos\theta}$$

Now for $\theta \in [\pi/2, \pi]$, $\cos\theta \leqslant 1 - 2\theta/\pi$ and so:

$$e^{xR\cos\theta} \leqslant e^{xR - 2xR\theta/\pi},$$

So,

$$\begin{aligned}
I_{BC} &\leqslant \frac{Re^{xR}}{|R-|a||\sqrt{|R^2-b^2|}} \int_{\frac{\pi}{2}}^{\pi} e^{-2Rx\theta/\pi} d\theta \\
&= \frac{Re^{xR}}{|R-|a||\sqrt{|R^2-b^2|}} \left[-\frac{\pi}{2Rx} e^{-2Rx\theta/\pi} \right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{\pi e^{xR}}{2x|R-|a||\sqrt{|R^2-b^2|}} (e^{-Rx} - e^{-2Rx}) \\
&= \frac{\pi}{2x|R-|a||\sqrt{|R^2-b^2|}} (1 - e^{-Rx})
\end{aligned}$$

This clearly tends to zero as $R \to \infty$ and so $I_{BC} = 0$ as $R \to \infty$. A similar estimation may be performed for I_{HA} but the inequality $\cos\theta \leqslant 2\theta/\pi - 3$ is used and all the estimations are the same. The next integral to examine is I_γ, using the contour $s = b + \varepsilon e^{i\theta}$, $ds = i\varepsilon e^{i\theta} d\theta$. The estimates are then:

$$|s-a| \geqslant ||b-a|-\varepsilon|, \quad |\sqrt{s^2-b^2}| \geqslant \sqrt{\varepsilon}|2|b|-\varepsilon|$$

The exponential becomes:

$$|e^{ts}| = t^{tb} e^{t\varepsilon\cos\theta} \leqslant e^{tb} e^{t\varepsilon}$$

So the integral becomes:

$$|I_\gamma| \leqslant \frac{2\pi e^{tb} e^{t\varepsilon}\sqrt{\varepsilon}}{||b-a|-\varepsilon|\sqrt{|2|b|-\varepsilon|}} = O(\sqrt{\varepsilon})$$

So $|I_\gamma| \to 0$ as $\varepsilon \to 0$. The integrals over the contours Γ_u and Γ_l can be estimated in the same way to get the same estimate and so I_{Γ_u}, $I_{\Gamma_l} \to 0$ as $\varepsilon \to 0$.

The next integral to examine to $I_{D'E}$. Caution must be applied to this as as the phase of the branch cut must be examined in detail. The function in question is $\sqrt{s^2-b^2} = \sqrt{(s-b)(s+b)}$. Denoting for convenience $s - b = r_1 e^{i\theta}$ and $s + b = r_2 e^{i\phi}$, the arguments of the factors are $arg(s-a) = \pi$ and $arg(s+b) = 0$, and so the phase of the square root is $\pi/2$. Writing the path as $s = u$, where $-b + \varepsilon \leqslant u \leqslant b - \varepsilon$, the integral becomes:

$$I_{D'E} = \int_{-b+\varepsilon}^{b-\varepsilon} \frac{e^{xu}}{(u-a)e^{i\pi/2}\sqrt{u^2-b^2}} du$$

Taking the limit as $\varepsilon \to 0$ shows that:

$$I_{D'E} = -i \int_{-b}^{b} \frac{e^{xu}}{(u-a)\sqrt{u^2-b^2}} du$$

The other integral of this type is $I_{FG'}$, once again the path is $s = u$ with $b - \varepsilon \leqslant u \leqslant -b + \varepsilon$ and this time the arguments of the branches of the square root is different, $arg(s+b)=2\pi$ and $arg(s+b)=\pi$, so the total phase of the square root is $\exp(3\pi i/2)$, so $I_{FG'}$ is:

$$I_{FG'} = \int_{b-\varepsilon}^{-b+\varepsilon} \frac{e^{xu}}{(u-a)e^{3\pi i/2}\sqrt{u^2-b^2}}$$

when $\varepsilon \to 0$ the integral becomes:

$$I_{FG'} = i\int_{b}^{-b} \frac{e^{xu}}{(u-a)\sqrt{u^2-b^2}}du = -i\int_{-b}^{b} \frac{e^{xu}}{(u-a)\sqrt{u^2-b^2}}du$$

So the two integral are the same. For the integrals I_{CD} and I_{GH}, the arguments for $s-b$ and $s+b$ for the path CD and GH are the same and so the overall phase will be the same for the square root. As the two path are equal an opposite, the integrals will cancel when the limits are taken, so:

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} I_{CD} + I_{GH} = 0$$

The final integral is the inverse Laplace transform:

$$I_{AB} = \int_{\sigma-Ri}^{\sigma+Ri} \frac{e^{xs}}{(s-a)\sqrt{s^2-b^2}}ds,$$

and the final result becomes:

$$\int_{\sigma-Ri}^{\sigma+Ri} \frac{e^{xs}}{(s-a)\sqrt{s^2-b^2}}ds - 2i\int_{-b}^{b} \frac{e^{xu}}{(u-a)\sqrt{u^2-b^2}}du = 2\pi i\frac{e^{at}}{\sqrt{a^2-b^2}}$$

So the inverse Laplace transform is:

$$\frac{1}{2\pi i}\int_{\sigma-Ri}^{\sigma+Ri} \frac{e^{xs}}{(s-a)\sqrt{s^2-b^2}}ds = \frac{1}{\pi}\int_{-b}^{b} \frac{e^{xu}}{(u-a)\sqrt{u^2-b^2}}du + \frac{e^{at}}{\sqrt{a^2-b^2}}$$