(a) The potential on the axis (chosen as the z-axis) of a uniformly charged ring of charge Q and radius R is

$$\psi(z) = \frac{Q}{\sqrt{z^2 + R^2}} \tag{1}$$

because every point on the ring is the same distance $\sqrt{z^2 + R^2}$ from the point z.

(b) We can write the potential in terms of the variables r and θ as

$$\psi(r, 0^\circ) = \frac{Q}{\sqrt{r^2 + R^2}}. \tag{2}$$

This can be expanded for $r > R$ as

$$\psi(r, 0^\circ) = Q[r^2 + R^2]^{-1/2} = \frac{Q}{r}\left[1 + \frac{R^2}{r^2}\right]^{-1/2} = Q\sum_{n=0}^{\infty}\begin{pmatrix} -1/2 \\ n \end{pmatrix}\frac{R^{2n}}{r^{2n+1}}, \quad r > R. \tag{3}$$

We introduce the index $l = 2n$. Then l must be even and $n = l/2$. The expansion for $\psi(r, 0^\circ)$ can be written as

$$\psi(r, 0^\circ) = Q\sum_{\text{even } l}\begin{pmatrix} -1/2 \\ l/2 \end{pmatrix}\frac{R^l}{r^{l+1}}, \quad r > R. \tag{4}$$

This can now be extended to all angles by multiplying each term in the sum by $P_l(\cos\theta)$:

$$\psi(r, \theta) = Q\sum_{\text{even } l}\begin{pmatrix} -1/2 \\ l/2 \end{pmatrix}\frac{R^l P_l(\cos\theta)}{r^{l+1}}, \quad r > R. \tag{5}$$

For $r < R$ the potential still satifies Laplace's equation because there is no charge inside the ring. We can expand $\psi(r, 0^\circ)$ in powers of r^2/R^2, and following steps similar to those above leads to

$$\psi(r, \theta) = Q\sum_{\text{even } l}\begin{pmatrix} -1/2 \\ l/2 \end{pmatrix}\frac{r^l P_l(\cos\theta)}{R^{l+1}}, \quad r < R. \tag{6}$$