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Criteria for the reality of matrix eigenvalues

By

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In this note we are, essentially, concerned with generalizations of the (known) fact that an $n \times n$ matrix with n linearly independent eigenvectors all corresponding to real eigenvalues is similar to a hermitian matrix, and can consequently be transformed into its conjugate transpose by a positive definite hermitian similarity. We first establish, for any positive integer m, an analogous necessary and sufficient condition that a given square complex matrix A should have a set of real eigenvalues, not necessarily all distinct, to which there correspond at least m linearly independent eigenvectors; this of course implies a corresponding result about pure imaginary eigenvalues. We also obtain an analogous result concerning eigenvalues of modulus unity.

As a simple application of our more general results, we establish, in Theorem 4, the reality of the eigenvalues of a certain rather special type of matrix. Throughout, we shall use $A = (a_{ij})$ to denote an arbitrary $n \times n$ complex matrix, and A^* will denote the transposed conjugate matrix; we denote the rank of A by $r(A)$.

Theorem 1. *A necessary*[1]) *and sufficient condition for the existence of a set of m linearly independent eigenvectors of A all corresponding to real eigenvalues (or, equivalently, for the Jordan normal form of A to have at least m blocks with real eigenvalues) is that there exist a positive semi-definite hermitian matrix S of rank m such that $A S$ is hermitian, i.e.*

$$A S = S A^* . \tag{1}$$

Proof of necessity. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ be arbitrary given eigenvectors of A corresponding to real eigenvalues $\alpha_1, \ldots, \alpha_m$ (which we do not assume distinct), and define $n \times m$, $m \times m$ matrices

$$X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m), \qquad D = \operatorname{diag}(\alpha_1, \ldots, \alpha_m).$$

Then $A X = X D$, and so, since D is real,

$$A X X^* = X D X^* = X (X D)^* = X (A X)^* = X X^* A,$$

i.e. we can certainly satisfy (1) by taking $S = X X^*$, which is plainly a positive semi-definite hermitian matrix.

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[1]) Necessity is (as has been pointed out above) already known in the case $m = n$; see e.g. Reid [2], or, for a special case, Taussky [3]. For sufficiency, cf. also Turnbull and Aitken [4], p. 108.

Also, if any given linear combination of rows of S vanishes, say $\boldsymbol{y}S=0$ for some complex row n-vector $\boldsymbol{y}$, then $(\boldsymbol{y}X)(\boldsymbol{y}X)^*=\boldsymbol{y}S\boldsymbol{y}^*=0$, whence $\boldsymbol{y}X=0$, and, of course, conversely. Thus the row rank of S coincides with that of X, i.e. $r(S)=r(X)$.[2])

Finally, if the $\boldsymbol{x}_i$ are independent, then $r(X)=m$, and so $r(S)=m$, as required.

Proof of sufficiency. We apply an argument used by GODDARD and SCHNEIDER [*1*] in another connection (however, their result does not overlap ours). Given any positive semi-definite matrix S of rank m, we can find a non-singular matrix P such that $S=PNP^*$, where $N=\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ and I_m denotes the unit matrix of order m. Then (1) becomes

$$P^{-1}APN=N(P^{-1}AP)^*,$$

and so, partitioning $B=P^{-1}AP$ conformally with N, say $B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$,

we have
$$\begin{pmatrix} B_{11} & 0 \\ B_{21} & 0 \end{pmatrix} = \begin{pmatrix} B_{11}^* & B_{21}^* \\ 0 & 0 \end{pmatrix},$$

i.e.
$$P^{-1}AP=B=\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

where B_{11} is a hermitian matrix of order m, and consequently has a linearly independent set $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ of eigenvectors with real eigenvalues.[3]) Then clearly the n-vectors $P\begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{0} \end{pmatrix}, \ldots, P\begin{pmatrix} \boldsymbol{z}_m \\ \boldsymbol{0} \end{pmatrix}$ (with $n-m$ zeroes below the $\boldsymbol{z}_i$) form a linearly independent set of m eigenvectors for A with (the same set of) real eigenvalues, as required.

We note two immediate corollaries of Theorem 1:

COROLLARY 1. *If A has m distinct real eigenvalues, then there is a positive semi-definite hermitian matrix S of rank m satisfying* (1); *conversely, given any such S, then A must have at least m (not necessarily distinct) real eigenvalues, and, if $m=n$, then A is diagonable.*

COROLLARY 2. *If A has real trace or real non-zero determinant, and we are given* (1) *with $r(S)=n-1$, then all the eigenvalues of A are real.*

Of these, the first follows trivially from the theorem. As for the second, we have only to remark that A has at least $n-1$ real eigenvalues by Theorem 1, and so, since the trace and determinant of a matrix are respectively just the sum and product of its eigenvalues, it follows that the n-th eigenvalue must be real also. More generally, as has been kindly pointed out to us by K. GOLDBERG, it is enough that, in the characteristic function of A, there occurs some consecutive pair of real coefficients with the higher non-zero.

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[2]) That $r(XX^*)=r(X)$, even for arbitrary X, is of course well known, but we have included the argument for completeness.

[3]) Since P can be computed from S by the use of a standard algorithm, we can always compute the B_{ij} effectively when S is given numerically.

Another immediate consequence of Theorem 1 (obtained by replacing A by iA) is the following:

THEOREM 2. *A necessary and sufficient condition for the existence of a set of m linearly independent eigenvectors of A all corresponding to purely imaginary eigenvalues is that there exist a positive semi-definite hermitian matrix S of rank m such that AS is skew hermitian, i.e. $AS = -SA^*$.*

The following somewhat analogous result does not seem to be deducible directly from Theorem 1, but can be proved by closely parallel arguments:

THEOREM 3. *A necessary and sufficient condition for the existence of a set of m linearly independent eigenvectors of A all corresponding to eigenvalues of absolute value 1 is that there exist a positive semi-definite hermitian matrix S of rank m such that*

$$ASA^* = S. \tag{2}$$

Proof of necessity. Defining X, D as in the proof of necessity in Theorem 1, but with the α_i now assumed to be of unit modulus rather than real, we still have $AX = XD$, so that

$$AXX^*A^* = AX(AX)^* = XD(XD)^* = X(DD^*)X^* = XX^*;$$

thus we see, as before, that the choice $S = XX^*$ satisfies all the required conditions.

Proof of sufficiency. Given S of rank m, then, with N, P as in our proof sufficiency in Theorem 1 above, we may rewrite (2) in the form

$$P^{-1}APN(P^{-1}AP)^* = N,$$

i.e., on defining $B = P^{-1}AP$ and partitioning as before,

$$\begin{pmatrix} B_{11}B_{11}^* & B_{11}B_{21}^* \\ B_{21}B_{11}^* & B_{21}B_{21}^* \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $B_{21} = 0$ and $P^{-1}AP = B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ with B_{11} unitary, and so the result follows as before.

We can of course deduce results analogous to Corollary 1 for Theorems 2 and 3; the same is true also of Corollary 2 if we omit the reference to the trace function for the latter theorem.

We may use the results we have obtained to establish the reality of the eigenvalues of more restricted classes of matrix. For example:

THEOREM 4. *Given $A = (a_{ij})$, let $r_i = \sum_{k=1}^{n} a_{ik}$ $(i = 1, \ldots, n)$, define an $n \times n$ matrix $E = (e_{ij})$ by taking $e_{ij} = r_i - \overline{r}_j$, and suppose that*

$$A^* - A = cE \tag{3}$$

for some real $c > -1/n$ (or $c \geqq -1/n$ if A has real trace or real non-zero determinant). Then A has all its eigenvalues real (and A is diagonable if $c \neq -1/n$).

Proof. If J denotes the $n \times n$ matrix consisting entirely of 1's, then clearly $E = AJ - JA^*$, so that (3) may be rewritten in the form

$$A(I + cJ) = (I + cJ)A^*.$$

Hence, taking $S = I + cJ$, which is easily checked to be a positive semi-definite hermitian matrix of rank at least $n-1$ for all $c \geqq -1/n$ (and positive definite for $c > -1/n$), we obtain the result at once from Theorem 1 and Corollary 2 above.

It is clear from the argument of Theorem 4 that it would be very easy to concoct any number of results along similar lines; but this single example should suffice to illustrate the principle. The case $c = 0$ of Theorem 4 is of course standard; we note also, in connection with the final parenthetic clause of the conclusion, that even a real A need not be diagonable when $c = -1/n$, as the example $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ shows.

References

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