Numerical Solution of the Diffusion Equation in Cylindrical Co-ordinates via the Crank-Nicholson Method

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The diffusion equation in spherical polar co-ordinates is:

$$\theta\frac{\partial T}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+Q(t) \tag{1}$$

With initial condition:

$$T(0,r)=0 \tag{2}$$

and boundary conditions:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0,\quad \left.\frac{\partial T}{\partial r}\right|_{r=R}=-hT(t,R) \tag{3}$$

The method will be Crank-Nicholsen method:

$$\begin{aligned}\theta\frac{T_{i+1,j}-T_{i,j}}{\delta t}=\frac{1}{2}\Bigg[&\frac{\kappa}{r_j}\frac{T_{i+1,j+1}-T_{i+1,j-1}}{\delta r}+\kappa\frac{T_{i+1,j+1}-2T_{i+1,j}+T_{i+1,j-1}}{\delta r^2}+Q_{i+1}+\\&+\frac{\kappa}{r_j}\frac{T_{i,j+1}-T_{i,j-1}}{\delta r}+\kappa\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\delta r^2}+Q_i\Bigg]\end{aligned} \tag{4}$$

This can be arranged in the following way:

$$\begin{aligned}&\left(A+\frac{B}{r_j}\right)T_{i+1,j+1}-(2A+\theta)T_{i+1,j}+\left(A-\frac{B}{r_j}\right)T_{i+1,j-1}=\\&=-\left(A+\frac{B}{r_j}\right)T_{i,j+1}+(2A-\theta)T_{i,j}-\left(A-\frac{B}{r_j}\right)T_{i,j-1}-\frac{1}{2}(Q_{i+1}+Q_i)\delta t\end{aligned} \tag{5}$$

Where:

$$A=\frac{\kappa\delta t}{2\delta r^2},\quad B=\frac{\kappa\delta t}{2\delta r} \tag{6}$$

To examine the outer boundary condition. Setting $j=N$ shows that:

$$\begin{aligned}&\left(A+\frac{B}{r_N}\right)T_{i+1,N+1}-(2A+\theta)T_{i+1,N}+\left(A-\frac{B}{r_N}\right)T_{i+1,N-1}=\\&=-\left(A+\frac{B}{r_N}\right)T_{i,N+1}+(2A-\theta)T_{i,N}-\left(A-\frac{B}{r_N}\right)T_{i,N-1}-\frac{1}{2}(Q_{i+1}+Q_i)\delta t\end{aligned} \tag{7}$$

To get rid off the $N+1$ term in the spacial component, use the boundary condition. The discrete version of this condition is:

$$\frac{T_{i,j+1} - T_{i,j-1}}{2\delta r} = -hT_{i,j} \Rightarrow T_{i,N+1} = T_{i,N-1} - 2h\delta r T_{i,N} \tag{8}$$

Inserting this expression shows that:

$$\begin{aligned} &\left(A + \frac{B}{r_N}\right)(T_{i+1,N-1} - 2h\delta r T_{i+1,N}) - (2A+\theta)T_{i+1,N} + \left(A - \frac{B}{r_N}\right)T_{i+1,N-1} = \\ &= -\left(A + \frac{B}{r_N}\right)(T_{i,N-1} - 2h\delta r T_{i,N}) + (2A-\theta)T_{i,N} - \left(A - \frac{B}{r_N}\right)T_{i,N-1} - \\ &\qquad - \frac{1}{2}(Q_{i+1} + Q_i)\delta t \end{aligned} \tag{9}$$

Rearranging this to obtain:

$$\begin{aligned} &-\left[2h\delta r\left(A + \frac{B}{r_N}\right) + 2A + \theta\right]T_{i+1,N} + 2AT_{i+1,N-1} = \\ &= \left[2A - \theta + 2h\delta r\left(A + \frac{B}{r_N}\right)\right]T_{i,N} - 2AT_{i,N-1} - \frac{1}{2}(Q_{i+1} + Q_i)\delta t \end{aligned} \tag{10}$$

To obtain the boundary condition at $r = 0$, take the limit as $r \to 0$, this makes:

$$\lim_{r \to 0} \frac{\frac{\partial T}{\partial r}}{r} = \frac{\partial^2 T}{\partial r^2}$$

This makes the equation:

$$\theta \frac{\partial T}{\partial t} = 2\kappa \frac{\partial^2 T}{\partial r^2} + Q(t) \tag{11}$$

The process is then continued in the Crank-Nicholsen way:

$$\begin{aligned} \theta \frac{T_{i+1,1} - T_{i,1}}{\delta t} = \frac{1}{2}\Bigg[&2\kappa \frac{T_{i+1,2} - 2T_{i+1,1} + T_{i+1,0}}{\delta r^2} + Q_{i+1} + \\ &+ 2\kappa \frac{T_{i,2} - 2T_{i,1} + T_{i,0}}{\delta r^2} + Q_i\Bigg] \end{aligned} \tag{12}$$

Which can be arranged to:

$$\begin{aligned} &2AT_{i+1,2} - (4A - \theta)T_{i+1,1} + 2AT_{i+1,0} = \\ &\qquad = -2AT_{i,2} - (4A + \theta)T_{i,1} - 2AT_{i,0} - \frac{1}{2}(Q_{i+1} + Q_i)\delta t \end{aligned} \tag{13}$$

Finally the boundary condition is used to show that $T_{i,0} = T_{i,2}$, this makes:

$$4AT_{i+1,2} - (4A - \theta)T_{i+1,1} = -4AT_{i,2} - (4A + \theta)T_{i,1} - \frac{1}{2}(Q_{i+1} + Q_i)\delta t \tag{14}$$

The equation can be written as:

$$\alpha[T]^{i+1} = \beta[T]^i + \gamma \tag{15}$$

For 6 points the matrices α, β and γ are:

$$\alpha = \begin{pmatrix} \theta - 4A & 4A & 0 & 0 & 0 & 0 \\ A - \frac{B}{r_2} & -(2A+\theta) & A + \frac{B}{r_2} & 0 & 0 & 0 \\ 0 & A - \frac{B}{r_3} & -(2A+\theta) & A + \frac{B}{r_3} & 0 & 0 \\ 0 & 0 & A - \frac{B}{r_4} & -(2A+\theta) & A + \frac{B}{r_4} & 0 \\ 0 & 0 & 0 & A - \frac{B}{r_5} & -(2A+\theta) & A + \frac{B}{r_5} \\ 0 & 0 & 0 & 0 & 2A & a_1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} -(\theta + 4A) & -4A & 0 & 0 & 0 & 0 \\ -\left(A - \frac{B}{r_2}\right) & 2A - \theta & -\left(A + \frac{B}{r_2}\right) & 0 & 0 & 0 \\ 0 & -\left(A - \frac{B}{r_3}\right) & 2A - \theta & -\left(A + \frac{B}{r_3}\right) & 0 & 0 \\ 0 & 0 & -\left(A - \frac{B}{r_4}\right) & 2A - \theta & -\left(A + \frac{B}{r_4}\right) & 0 \\ 0 & 0 & 0 & -\left(A - \frac{B}{r_5}\right) & 2A - \theta & -\left(A + \frac{B}{r_5}\right) \\ 0 & 0 & 0 & 0 & -2A & a_2 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \\ -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \\ -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \\ -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \\ -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \\ -\frac{1}{2}(Q_{i+1} + Q_i)\delta t \end{pmatrix}$$

where:

$$a_1 = -\left(2h\delta r\left(A + \frac{B}{r_6}\right) + 2A + \theta\right), \quad a_2 = 2A - \theta + 2h\delta r\left(A + \frac{B}{r_6}\right) \tag{16}$$