



Using the first equation show the second. Show all of the steps. Report results that can be pasted into MS word.

Thought for 1m 48s

Here's a clean, copy-paste friendly derivation that starts from

$$\frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\psi(x, t)|^2 dx$$

and ends at the standard result

$$\frac{d\langle x \rangle}{dt} = \frac{\hbar}{2mi} \int_{-\infty}^{\infty} [\psi^*(x, t) \partial_x \psi(x, t) - \psi(x, t) \partial_x \psi^*(x, t)] dx = \frac{\hbar}{m} \text{Im} \int_{-\infty}^{\infty} \psi^* \partial_x \psi dx.$$

Assumptions:  $V(x, t)$  is real,  $\psi$  and its first derivative vanish at  $|x| \rightarrow \infty$  fast enough that surface terms are zero, and  $\psi$  satisfies the TDSE.

**Step 1. Differentiate  $\langle x \rangle$ .**

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx \quad \Rightarrow \quad \frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} (\psi^* \psi) dx = \int_{-\infty}^{\infty} x \left( \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \right) dx.$$

**Step 2. Use the time-dependent Schrödinger equation.**

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \partial_x^2 \psi + V \psi \quad \Rightarrow \quad \frac{\partial \psi}{\partial t} = \frac{1}{i\hbar} \left( -\frac{\hbar^2}{2m} \partial_x^2 \psi + V \psi \right).$$

Taking the complex conjugate,

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \partial_x^2 \psi^* + V \psi^* \quad \Rightarrow \quad \frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar} \left( -\frac{\hbar^2}{2m} \partial_x^2 \psi^* + V \psi^* \right).$$

**Step 3. Substitute and cancel the potential terms.**

$$\begin{aligned}\frac{d\langle x \rangle}{dt} &= \int x [\psi^* \frac{1}{i\hbar} (-\frac{\hbar^2}{2m} \partial_x^2 \psi + V \psi) + \psi (-\frac{1}{i\hbar} (-\frac{\hbar^2}{2m} \partial_x^2 \psi^* + V \psi^*))] dx \\ &= \frac{\hbar}{2mi} \int x [\psi \partial_x^2 \psi^* - \psi^* \partial_x^2 \psi] dx \quad (\text{since } \psi^* V \psi - \psi V \psi^* = 0).\end{aligned}$$

**Step 4. Integrate by parts to remove second derivatives.**

For the first term,

$$\int x \psi \partial_x^2 \psi^* dx = [x \psi \partial_x \psi^*]_{-\infty}^{\infty} - \int (\partial_x(x\psi)) \partial_x \psi^* dx = [\dots]_{-\infty}^{\infty} - \int (\psi + x \partial_x \psi) \partial_x \psi^* dx.$$

For the second term,

$$\int x \psi^* \partial_x^2 \psi dx = [x \psi^* \partial_x \psi]_{-\infty}^{\infty} - \int (\psi^* + x \partial_x \psi^*) \partial_x \psi dx.$$

Subtracting the two expressions, the  $x \partial_x \psi \partial_x \psi^*$  terms cancel identically, and the boundary terms vanish by the decay assumption on  $\psi$ . Thus

$$\int x [\psi \partial_x^2 \psi^* - \psi^* \partial_x^2 \psi] dx = - \int (\psi \partial_x \psi^* - \psi^* \partial_x \psi) dx.$$

**Step 5. Final form.**

$$\frac{d\langle x \rangle}{dt} = \frac{\hbar}{2mi} \int [\psi^* \partial_x \psi - \psi \partial_x \psi^*] dx = \frac{\hbar}{m} \text{Im} \int \psi^* \partial_x \psi dx = \frac{1}{m} \langle p \rangle,$$

since  $\langle p \rangle = \int \psi^* (-i\hbar \partial_x) \psi dx$ .

(Equivalently:  $\frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{2m} \int [\psi^* \partial_x \psi - \psi \partial_x \psi^*] dx$ .)